

LIFTABLE VECTOR FIELDS OVER CORANK ONE MULTIGERMS

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ABSTRACT. In this paper, a systematic method is given to construct all liftable vector fields over an analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ of corank at most one admitting a one-parameter stable unfolding.

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Key words: liftable vector field, reduced Kodaira-Spencer-Mather map, higher version of the reduced Kodaira-Spencer-Mather map, finite multiplicity, corank at most one.

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1. INTRODUCTION

Let S be a finite subset of \mathbb{K}^n , where \mathbb{K} is the real field \mathbb{R} or the complex field \mathbb{C} and n is a positive integer. A map-germ $(\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ is called a *multigerm*, and it is called a *mono-germ* if S consists of only one point. Let C_S (resp., C_0) be the set of analytic (that is, real-analytic if $\mathbb{K} = \mathbb{R}$ or holomorphic if $\mathbb{K} = \mathbb{C}$) multigerms of function $(\mathbb{K}^n, S) \rightarrow \mathbb{K}$ (resp., germs of function $(\mathbb{K}^p, 0) \rightarrow \mathbb{K}$), and let m_S (resp., m_0) be the subset of C_S (resp., C_0) consisting of analytic function-germs $(\mathbb{K}^n, S) \rightarrow (\mathbb{K}, 0)$ (resp., $(\mathbb{K}^p, 0) \rightarrow (\mathbb{K}, 0)$). It is clear that the sets C_S and C_0 have natural \mathbb{K} -algebra structures induced by the \mathbb{K} -algebra structure of \mathbb{K} . For an analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, let $f^* : C_0 \rightarrow C_S$ be the \mathbb{K} -algebra homomorphism defined by $f^*(u) = u \circ f$. Set $Q(f) = C_S / f^* m_0 C_S$. A multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ is said to have *finite multiplicity* if $Q(f)$ is a finite dimensional \mathbb{K} -vector space. It is well-known that if a multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ has finite multiplicity, then n must be less than or equal to p .

For an analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ such that $f(S) \subset T$, where S (resp., T) is a finite subset of \mathbb{K}^n (resp., \mathbb{K}^p), let $\theta_S(f)$ be the C_S -module consisting of germs of analytic vector fields along f . We may identify $\theta_S(f)$ with $\underbrace{C_S \oplus \cdots \oplus C_S}_p$. We set $\theta_S(n) = \theta_S(id_{(\mathbb{K}^n, S)})$ and $\theta_0(p) = \theta_{\{0\}}(id_{(\mathbb{K}^p, 0)})$, where $id_{(\mathbb{K}^n, S)}$ (resp., $id_{(\mathbb{K}^p, 0)}$) is the germ of the identity mapping of (\mathbb{K}^n, S) (resp., $(\mathbb{K}^p, 0)$).

For a given analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, following Mather ([14]), we define tf and ωf as follows:

$$\begin{aligned} tf : \theta_S(n) &\rightarrow \theta_S(f), & tf(\eta) &= df \circ \eta, \\ \omega f : \theta_0(p) &\rightarrow \theta_S(f), & \omega f(\xi) &= \xi \circ f, \end{aligned}$$

where df is the differential of f . For f , following Wall ([28]), we set

$$\begin{aligned} T\mathcal{R}(f) &= tf(m_S \theta_S(n)), & T\mathcal{R}_e(f) &= tf(\theta_S(n)), \\ T\mathcal{L}(f) &= \omega f(m_0 \theta_0(p)), & T\mathcal{L}_e(f) &= \omega f(\theta_0(p)), \\ T\mathcal{A}(f) &= T\mathcal{R}(f) + T\mathcal{L}(f), & T\mathcal{A}_e(f) &= T\mathcal{R}_e(f) + T\mathcal{L}_e(f), \\ TK(f) &= T\mathcal{R}(f) + f^* m_0 \theta_S(f), & TK_e(f) &= T\mathcal{R}_e(f) + f^* m_0 \theta_S(f). \end{aligned}$$

For a given analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, following Arnol'd ([1]), we call a vector field $\xi \in \theta_0(p)$ *liftable over f* if $\xi \circ f$ belongs to $T\mathcal{R}_e(f)$. The set of vector fields liftable over f is denoted by $Lift(f)$. It is clear that $Lift(f)$ naturally has a C_0 -module structure.

The use of liftable vector fields has proven to be a fundamental tool in the study of classification techniques. In [1] and [6], and more recently in [5], [10] and [22], $Lift(f)$ has played a central role in the development of operations and in order to calculate the codimensions of the multigerms resulting from these operations. However, in general, obtaining generators of $Lift(f)$ can be a very hard task. In fact, some articles such as [11] are devoted to constructing $Lift(f)$ for a particular case of germs.

The purpose of this paper is giving a systematic method to construct liftable vector fields over a multigerm. In order to create the systematic method, we first concentrate on obtaining a reasonable class of multigerms for which the following problems can be affirmatively answered.

Problem 1. Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be an analytic multigerm.

- (1) Is the module of vector fields liftable over f finitely generated ?
- (2) Can we characterize the minimal number of generators when the module of vector fields liftable over f is finitely generated ?
- (3) Can we calculate the minimal number of generators when the module of vector fields liftable over f is finitely generated ?
- (4) Can we construct generators when the module of vector fields liftable over f is finitely generated ?

In order to obtain such a reasonable class, we generalize Mather's homomorphism ([15])

$$\bar{\omega}f : \frac{\theta_0(p)}{m_0\theta_0(p)} \rightarrow \frac{\theta_S(f)}{TK_e(f)}$$

defined by $\bar{\omega}f([\xi]) = [\omega f(\xi)]$. Notice that

$$\frac{\theta_S(f)}{TK_e(f)} \cong \frac{\frac{\theta_S(f)}{TR_e(f)}}{f^*m_0\left(\frac{\theta_S(f)}{TR_e(f)}\right)}$$

as finite dimensional vector spaces over \mathbb{K} for any analytic multigerm f satisfying $\dim_{\mathbb{K}} \theta_S(f)/TK_e(f) < \infty$. Thus, by the preparation theorem (for instance, see [2]), we have that $\theta_S(f) = T\mathcal{A}_e(f)$ if and only if $\bar{\omega}f$ is surjective for any analytic multigerm f satisfying $\dim_{\mathbb{K}} \theta_S(f)/TK_e(f) < \infty$. In the case that $\mathbb{K} = \mathbb{C}$, $n \geq p$ and $S = \{\text{one point}\}$, the map $\hat{\omega}f : \theta_0(p) \rightarrow \frac{\theta_S(f)}{TR_e(f)}$ given by $\hat{\omega}f(\xi) = [\omega f(\xi)]$ is called the *Kodaira-Spencer map* of f and Mather's homomorphism $\bar{\omega}f$ is called the *reduced Kodaira-Spencer map* of f ([13]). Thus, $\bar{\omega}f$, which we call the *reduced Kodaira-Spencer-Mather map*, is a generalization of the reduced Kodaira-Spencer map of f ; and the module of vector fields liftable over f is the kernel of $\hat{\omega}f$. We would like to have higher versions of $\bar{\omega}f$. For a non-negative integer i , an element of m_S^i or m_0^i is a germ of analytic function such that the terms of the Taylor series of it up to $(i-1)$ are zero. Thus, $m_S^0 = C_S$ and $m_0^0 = C_0$. For any non-negative integer i and a given analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, we let

$${}_i\bar{\omega}f : \frac{m_0^i\theta_0(p)}{m_0^{i+1}\theta_0(p)} \rightarrow \frac{f^*m_0^i\theta_S(f)}{TR_e(f) \cap f^*m_0^i\theta_S(f) + f^*m_0^{i+1}\theta_S(f)}$$

be a homomorphism of C_0 -modules via f defined by ${}_i\bar{\omega}f([\xi]) = [\omega f(\xi)]$. Then, ${}_i\bar{\omega}f$ is clearly well-defined. In this paper, we call ${}_i\bar{\omega}f$ a *higher version of reduced Kodaira-Spencer-Mather map*. Note that ${}_0\bar{\omega}f = \bar{\omega}f$. Similarly as the target module of $\bar{\omega}f$, for any non-negative integer i and any analytic multigerm f satisfying $\dim_{\mathbb{K}} \theta_S(f)/TK_e(f) < \infty$, the target module of ${}_i\bar{\omega}f$ is isomorphic to the following:

$$\frac{\frac{f^*m_0^i\theta_S(f)}{TR_e(f) \cap f^*m_0^i\theta_S(f)}}{f^*m_0\left(\frac{f^*m_0^i\theta_S(f)}{TR_e(f) \cap f^*m_0^i\theta_S(f)}\right)}.$$

Thus, again by the preparation theorem, we have that $f^*m_0^i\theta_S(f) \subset T\mathcal{A}_e(f)$ if and only if ${}_i\bar{\omega}f$ is surjective. The following clearly holds:

Lemma 1.1. *Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be an analytic multigerm satisfying the condition $\dim_{\mathbb{K}} \theta_S(f)/TK_e(f) < \infty$. Then, the following hold:*

- (1) Suppose that there exists a non-negative integer i such that $i\bar{\omega}f$ is surjective. Then, $j\bar{\omega}f$ is surjective for any integer j such that $i < j$.
- (2) Suppose that there exists a non-negative integer i such that $i\bar{\omega}f$ is injective. Then, $j\bar{\omega}f$ is injective for any non-negative integer j such that $i > j$.

Definition 1.1. Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be an analytic multigerm satisfying the condition $\dim_{\mathbb{K}} \theta_S(f)/TK_e(f) < \infty$.

- (1) Set $I_1(f) = \{i \in \{0\} \cup \mathbb{N} \mid i\bar{\omega}f \text{ is surjective}\}$. Define $i_1(f)$ as

$$i_1(f) = \begin{cases} \infty & (\text{if } I_1(f) = \emptyset) \\ \min I_1(f) & (\text{if } I_1(f) \neq \emptyset). \end{cases}$$

- (2) Set $I_2(f) = \{i \in \{0\} \cup \mathbb{N} \mid i\bar{\omega}f \text{ is injective}\}$. Define $i_2(f)$ as

$$i_2(f) = \begin{cases} -\infty & (\text{if } I_2(f) = \emptyset) \\ \max I_2(f) & (\text{if } \emptyset \neq I_2(f) \neq \{0\} \cup \mathbb{N}) \\ \infty & (\text{if } I_2(f) = \{0\} \cup \mathbb{N}). \end{cases}$$

An analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ is said to be *finitely determined* if there exists a positive integer k such that the inclusion $m_S^k \theta_S(f) \subset T\mathcal{A}_e(f)$ holds. The proof of the assertion (ii) of proposition 4.5.2 in [28] works well to show the following:

Proposition 1. Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be a finitely determined multigerm satisfying $\theta_S(f) \neq T\mathcal{A}_e(f)$. Then, $i_2(f) \geq 0$.

From here we concentrate on dealing with the case $n \leq p$ because the purpose of this paper is to construct liftable vector fields over a multigerm with finite multiplicity. However, the last section is devoted to extending the results to the case $n > p$. Suppose that $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ is finitely determined. Then, since it is clear that $f^*m_0C_S \subset m_S$, there exists a positive integer k such that the inclusion $f^*m_0^k\theta_S(f) \subset T\mathcal{A}_e(f)$ holds. Thus, $k\bar{\omega}f$ is surjective. Conversely, suppose that there exists a positive integer k such that $k\bar{\omega}f$ is surjective for an analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ satisfying $\dim_{\mathbb{K}} \theta_S(f)/TK_e(f) < \infty$. Then, as we have already confirmed, the inclusion $f^*m_0^k\theta_S(f) \subset T\mathcal{A}_e(f)$ holds by the preparation theorem. In the case $n \leq p$, by Wall's estimate (theorem 4.6.2 in [28]), the condition $\dim_{\mathbb{K}} \theta_S(f)/TK_e(f) < \infty$ implies that there exists an integer ℓ such that $m_S^\ell \subset f^*m_0C_S$. Hence, we have the following:

Proposition 2. Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be an analytic multigerm satisfying the condition $\dim_{\mathbb{K}} \theta_S(f)/TK_e(f) < \infty$. Suppose that $n \leq p$. Then, $i_1(f) < \infty$ if and only if f is finitely determined.

An analytic multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) is said to be of *corank at most one* if $\max\{n - \text{rank} Jf(s_j) \mid 1 \leq j \leq |S|\} \leq 1$ holds, where $Jf(s_j)$ is the Jacobian matrix of f at $s_j \in S$ and $|S|$ stands for the number of distinct points of S .

Proposition 3. Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) be a finitely determined multigerm of corank at most one. Then, $i_1(f) \geq i_2(f)$.

Proposition 3 is proved in §2. Proposition 3 yields the following corollary.

Corollary 1. Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) be a finitely determined multigerm of corank at most one. Suppose that there exists a non-negative integer i such that $i_1(f) = i_2(f) = i$. Then, the following hold:

- (1) For any non-negative integer j such that $j < i$, ${}_j\overline{\omega}f$ is injective but not surjective.
- (2) For any non-negative integer j such that $i < j$, ${}_j\overline{\omega}f$ is surjective but not injective.

Example 1.1. Let $e : \mathbb{K} \rightarrow \mathbb{K}^2$ be the embedding defined by $e(x) = (x, 0)$ and for any real number θ let $R_\theta : \mathbb{K}^2 \rightarrow \mathbb{K}^2$ be the linear map which gives the rotation of \mathbb{K}^2 about the origin with respect to the angle θ .

$$R_\theta \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

For any non-negative integer ℓ set $S = \{s_0, \dots, s_{\ell+1}\}$ ($s_j \neq s_k$ if $j \neq k$). Define $\theta_j = j \frac{\pi}{\ell+2}$ and set $e_j : (\mathbb{K}, s_j) \rightarrow (\mathbb{K}^2, 0)$ as $e_j(x_j) = R_{\theta_j} \circ e(x_j)$ for any j ($0 \leq j \leq \ell+1$), where $x_j = x - s_j$. Then, $E_\ell = \{e_0, \dots, e_{\ell+1}\} : (\mathbb{K}, S) \rightarrow (\mathbb{K}^2, 0)$ is a finitely determined multigerm of corank at most one. The image of E_ℓ is a line arrangement and hence the Euler vector field of the defining equation of the image of E_ℓ is a liftable vector field over E_ℓ . It follows that ${}_1\overline{\omega}E_\ell$ is not injective. Furthermore, it is easily seen that ${}_0\overline{\omega}E_\ell$ is injective even in the case $\ell = 0$ (in the case $\ell \geq 1$ this is a trivial corollary of Proposition 1). Thus, $i_2(E_\ell) = 0$. On the other hand, it is not hard to show that $i_1(E_\ell) = \ell$. Therefore, $i_1(E_\ell) - i_2(E_\ell) = \ell$.

This example shows that, in general, there are no upper bounds of $i_1(f) - i_2(f)$ for a finitely determined multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) of corank at most one. This example shows also that the integer $i_1(f) - i_2(f)$ measures how well-behaved a given finitely determined multigerm of corank at most one is from the viewpoint of liftable vector fields.

The following Theorem 1 shows that the desired reasonable class is the set consisting of finitely determined multigerms $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) of corank at most one satisfying $i_1(f) = i_2(f)$.

Theorem 1. *Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) be a finitely determined multigerm of corank at most one. Suppose that there exists a non-negative integer i such that $i_1(f) = i_2(f) = i$. Then, the minimal number of generators for the module of vector fields liftable over f is exactly $\dim_{\mathbb{K}} \ker({}_{i+1}\overline{\omega}f)$.*

Notice that the embedding e in Example 1.1 does not satisfy the assumption of Theorem 1. Actually, since ${}_0\overline{\omega}e$ is surjective but not injective, $i_1(e) = 0$ and $i_2(e) = -\infty$. On the other hand, the multigerm E_0 in Example 1.1 does satisfy the assumption of Theorem 1 though E_0 does not satisfy the assumption of Proposition 1. Furthermore, a lot of examples of Theorem 1 are given by Proposition 4 (see also Section 3).

Definition 1.2. (1) A multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ is said to be stable if it satisfies $\theta_S(f) = T_e\mathcal{A}(f)$.
 (2) Define the mapping $ev_0 : \theta_0(p) \rightarrow T_0(\mathbb{R}^p)$ by $ev_0(\eta) = \eta(0)$.
 (3) A stable multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ is said to be isolated if $ev_0(\eta) = 0$ for any $\eta \in \text{Lift}(f)$.

The following proposition shows that our reasonable class contains the set consisting of isolated stable multigerms $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) of corank at most one.

Proposition 4. *Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) be a finitely determined multigerm of corank at most one. Then, the following hold:*

- (1) In the case $i = 0$, the following hold:
 - (a) ${}_0\overline{\omega}f$ is surjective if and only if f is stable.
 - (b) ${}_0\overline{\omega}f$ is injective if and only if f is isolated.
- (2) In the case $i = 1$, the following hold:
 - (a) ${}_1\overline{\omega}f$ is surjective if and only if $T\mathcal{A}(f) = T\mathcal{K}(f)$.
 - (b) ${}_1\overline{\omega}f$ is injective if and only if for any $\eta \in \text{Lift}(f)$ η has no constant terms and no linear terms. Moreover, these equivalent conditions imply that $\dim_{\mathbb{K}} \theta_S(f)/T\mathcal{A}_e(f) > 1$.

Next, in order to answer (3) of Problem 1 for a given finitely determined multi-germ $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) of corank at most one such that $0 \leq i_1(f) = i_2(f) < \infty$, we generalize Wall's homomorphism ([28])

$$\overline{t}f : Q(f)^n \rightarrow Q(f)^p, \quad \overline{t}f([\eta]) = [tf(\eta)]$$

as follows. For a given analytic multigerms $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ satisfying the condition $\dim_{\mathbb{K}} Q(f) < \infty$, let $\delta(f)$ (resp., $\gamma(f)$) be the dimension of the vector space $Q(f)$ (resp., the dimension of the kernel of $\overline{t}f$). For the f and a non-negative integer i , we set ${}_iQ(f) = f^*m_0^i C_S / f^*m_0^{i+1} C_S$ and ${}_i\delta(f) = \dim_{\mathbb{K}} {}_iQ(f)$. Thus, we have that ${}_0Q(f) = Q(f)$ and ${}_0\delta(f) = \delta(f) = \dim_{\mathbb{K}} Q(f)$. The $Q(f)$ -modules ${}_iQ(f)^n$ and ${}_iQ(f)^p$ may be identified with the following respectively.

$$\frac{f^*m_0^i \theta_S(n)}{f^*m_0^{i+1} \theta_S(n)} \quad \text{and} \quad \frac{f^*m_0^i \theta_S(f)}{f^*m_0^{i+1} \theta_S(f)}.$$

Let ${}_i\gamma(f)$ be the dimension of the kernel of the following well-defined homomorphism of $Q(f)$ -modules.

$${}_i\overline{t}f : {}_iQ(f)^n \rightarrow {}_iQ(f)^p, \quad {}_i\overline{t}f([\eta]) = [tf(\eta)].$$

Then, we have that ${}_i\delta(f) < \infty$ if $\delta(f) < \infty$ and ${}_i\gamma(f) < \infty$ if $\gamma(f) < \infty$. For details on ${}_iQ(f)$, ${}_i\delta(f)$, ${}_i\overline{t}f$ and ${}_i\gamma(f)$, see [21].

Proposition 5. *Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be an analytic multigerms with finite multiplicity and of corank at most one. Suppose that there exists a non-negative integer i such that ${}_{i+1}\overline{\omega}f$ is surjective. Then, the following holds:*

$$\dim_{\mathbb{K}} \ker({}_{i+1}\overline{\omega}f) = p \cdot \binom{p+i}{i+1} - ((p-n) \cdot {}_{i+1}\delta(f) + {}_{i+1}\gamma(f) - {}_i\gamma(f)),$$

where the dot in the center stands for the multiplication.

Proposition 6. *Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be an analytic multigerms with finite multiplicity and of corank at most one. Then, the following hold:*

- (1) ${}_0\gamma(f) = \gamma(f) = \delta(f) - |S|$.
- (2)

$${}_i\delta(f) = \binom{n+i-1}{i} \cdot \delta(f), \quad {}_i\gamma(f) = \binom{n+i-1}{i} \cdot \gamma(f) \quad (i \in \mathbb{N} \cup \{0\}).$$

By combining Propositions 5 and 6, for an analytic multigerms f of corank at most one such that $\dim_{\mathbb{K}} Q(f) < \infty$, the \mathcal{A} -invariant “ $\dim_{\mathbb{K}} \ker({}_{i+1}\overline{\omega}f)$ ” can be calculated easily by using \mathcal{K} -invariants “ $\delta(f), \gamma(f)$ ” when there exists a non-negative integer i such that ${}_{i+1}\overline{\omega}f$ is surjective.

Theorem 2. *Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be an analytic multigerms with finite multiplicity and of corank at most one.*

- (1) Let $F : (\mathbb{K}^n \times \mathbb{K}^r, S \times \{0\}) \rightarrow (\mathbb{K}^p \times \mathbb{K}^r, (0, 0))$ be a stable unfolding of f . Let $\eta = (\eta_1, \dots, \eta_p, \eta_{p+1}, \dots, \eta_{p+r})$ be an element of the intersection $Lift(F) \cap Lift(g)$, where $g = \{g_1, \dots, g_r\}$ with $g_i : (\mathbb{K}^p \times \mathbb{K}^r, (0, 0)) \rightarrow (\mathbb{K}^p \times \mathbb{K}^r, (0, 0))$ defined by $g_i(X_1, \dots, X_p, \lambda_1, \dots, \lambda_r) = (X_1, \dots, X_p, \lambda_1, \dots, \lambda_{i-1}, \lambda_i^2, \lambda_{i+1}, \dots, \lambda_r)$ ($1 \leq i \leq r$). Then, $\bar{\eta}(X) = (\eta_1(X, 0), \dots, \eta_p(X, 0))$ is an element of $Lift(f)$.
- (2) Suppose that f admits a one-parameter stable unfolding $F : (\mathbb{K}^n \times \mathbb{K}, S \times \{0\}) \rightarrow (\mathbb{K}^p \times \mathbb{K}, (0, 0))$. Then, for any $\bar{\eta} \in Lift(f)$ there exists an element $\eta = (\eta_1, \dots, \eta_p, \eta_{p+1}) \in Lift(F) \cap Lift(g_1)$ such that the equality $\bar{\eta}(X) = (\eta_1(X, 0), \dots, \eta_p(X, 0))$ holds.

The proof of Theorem 1 provides a recipe for constructing all liftable vector fields over a finitely determined multigerm f of corank at most one satisfying $i_1(f) = i_2(f)$. In particular, by Proposition 4, all liftable vector fields over an isolated stable multigerm f of corank at most one can be constructed. Since any stable germ is \mathcal{A} -equivalent to a prism on an isolated stable multigerm, all liftable vector fields over any stable germ can be constructed. Moreover, by using Theorem 2, we can construct all liftable vector fields over an analytic multigerm f of corank at most one admitting a one-parameter stable unfolding F from $Lift(F)$. It is clear also that if f satisfies $\dim_{\mathbb{K}} \theta_S(f)/T\mathcal{A}_e(f) = 1$ (namely, f is a multigerm of \mathcal{A}_e -codimension one), then f admits a one-parameter stable unfolding. Thus, we can construct all liftable vector fields over a multigerm of corank at most one and of \mathcal{A}_e -codimension one. In particular, for any augmentation defined in [5], all liftable vector fields over it can be constructed by our recipe. In Remark 1 at the end of Section 6 an idea on how big the space of germs which admit a one-parameter stable unfolding is given.

This paper is organized as follows. In Section 2, proofs of Propositions 3, 4, 5, and 6 are given. In Section 3, examples for which actual calculations of minimal numbers of generators are carried out are given. Theorem 1 (resp., Theorem 2) is proved in Sections 4 (resp., Section 5). In Section 6, by constructing concrete generators for several examples using Theorem 2 and the proof of Theorem 1, it is explained in detail how to construct liftable vector fields over an analytic multigerm of corank at most one admitting a one-parameter stable unfolding. Finally, Section 7 generalizes the results for the case $n > p$.

2. PROOFS OF PROPOSITIONS 3, 4, 5, AND 6

Firstly, Proposition 6 is proved.

Proof of Proposition 6.

Set $S = \{s_1, \dots, s_{|S|}\}$ ($s_j \neq s_k$ if $j \neq k$) and for any j ($1 \leq j \leq |S|$) let f_j be the restriction $f|_{(\mathbb{K}^n, s_j)}$. Then, we have the following:

$$\begin{aligned} \delta(f) = \dim_{\mathbb{K}} Q(f) &= \sum_{j=1}^{|S|} \dim_{\mathbb{K}} Q(f_j) = \sum_{j=1}^{|S|} \delta(f_j). \\ \gamma(f) = \dim_{\mathbb{K}} \ker(\bar{t}f) &= \sum_{j=1}^{|S|} \dim_{\mathbb{K}} \ker(\bar{t}f_j) = \sum_{j=1}^{|S|} \gamma(f_j) \\ &= \sum_{j=1}^{|S|} (\delta(f_j) - 1) = \delta(f) - |S|. \end{aligned}$$

This completes the proof of the assertion 1 of Proposition 6.

Next we prove the assertion 2 of Proposition 6. Since f is of corank at most one, for any j ($1 \leq j \leq |S|$) there exist germs of diffeomorphism $h_j : (\mathbb{K}^n, s_j) \rightarrow (\mathbb{K}^n, s_j)$ and $H_j : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$ such that $H_j \circ f_j \circ h_j^{-1}$ has the following form:

$$H_j \circ f_j \circ h_j^{-1}(x, y) = (x, y^{\delta(f_j)} + f_{j,n}(x, y), f_{j,n+1}(x, y), \dots, f_{j,p}(x, y)).$$

Here, $(x, y) = (x_1, \dots, x_{n-1}, y)$ is the local coordinate with respect to the coordinate neighborhood (U_j, h_j) at s_j and $f_{j,q}$ satisfies $f_{j,q}(0, \dots, 0, y) = o(y^{\delta(f_j)})$ for any q ($n \leq q \leq p$). By the preparation theorem, C_{s_j} is generated by $1, y, \dots, y^{\delta(f_j)-1}$ as C_0 -module via f_j . Thus, $f_j^* m_0^i C_{s_j}$ is generated by elements of the following set as C_0 -module via f_j .

$$\left\{ x_1^{k_1} \dots x_{n-1}^{k_{n-1}} y^{k_n \delta(f_j) + \ell} \mid k_m \geq 0, \sum_{m=1}^n k_m = i, 0 \leq \ell \leq \delta(f_j) - 1 \right\}.$$

Thus, the following set is a basis of ${}_i Q(f_j)$.

$$\left\{ [x_1^{k_1} \dots x_{n-1}^{k_{n-1}} y^{k_n \delta(f_j) + \ell}] \mid k_m \geq 0, \sum_{m=1}^n k_m = i, 0 \leq \ell \leq \delta(f_j) - 1 \right\}.$$

Therefore, we have the following:

$$\begin{aligned} {}_i \delta(f) = \dim_{\mathbb{K}} {}_i Q(f) &= \sum_{j=1}^{|S|} \dim_{\mathbb{K}} {}_i Q(f_j) = \sum_{j=1}^{|S|} \dim_{\mathbb{K}} \frac{f_j^* m_0^i C_{s_j}}{f_j^* m_0^{i+1} C_{s_j}} \\ &= \sum_{j=1}^{|S|} \binom{n+i-1}{i} \cdot \delta(f_j) \\ &= \binom{n+i-1}{i} \cdot \sum_{j=1}^{|S|} \delta(f_j) \\ &= \binom{n+i-1}{i} \cdot \delta(f). \end{aligned}$$

Next we prove the formula for ${}_i \gamma(f)$. Since it is clear that ${}_i \gamma(f_j)$ does not depend on the particular choice of coordinate systems of (\mathbb{K}^n, s_j) and of $(\mathbb{K}^p, 0)$, we may

assume that f_j has the above form from the first. Then, it is easily seen that the following set is a basis of $\ker_i \bar{t}f_j$.

$$\left\{ \left[\underbrace{0 \oplus \cdots \oplus 0}_{(n-1) \text{ tuples}} \oplus x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} y^{k_n \delta(f_j) + \ell} \right] \mid k_m \geq 0, \sum_{m=1}^n k_m = i, 1 \leq \ell \leq \delta(f_j) - 1 \right\}.$$

Therefore, we have the following:

$$\begin{aligned} {}_i\gamma(f) &= \dim_{\mathbb{K}} \ker({}_i \bar{t}f) = \sum_{j=1}^{|S|} \dim_{\mathbb{K}} \ker({}_i \bar{t}f_j) = \sum_{j=1}^{|S|} \binom{n+i-1}{i} \cdot (\delta(f_j) - 1) \\ &= \binom{n+i-1}{i} \cdot \sum_{j=1}^{|S|} \gamma(f_j) \\ &= \binom{n+i-1}{i} \cdot \gamma(f). \end{aligned}$$

Q.E.D.

Secondly, Proposition 5 is proved.

Proof of Proposition 5

Consider the linear map ${}_{i+1}\bar{t}f$. Then, we have the following:

$$\dim_{\mathbb{K}} {}_{i+1}Q(f)^n = {}_{i+1}\gamma(f) + \dim_{\mathbb{K}} \text{Image}({}_{i+1}\bar{t}f).$$

Since $\dim_{\mathbb{K}} Q(f) < \infty$ and f is of corank at most one, it is easily seen that tf is injective. Hence we see that

$$\dim_{\mathbb{K}} \frac{T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f)}{T\mathcal{R}_e(f) \cap f^*m_0^{i+2}\theta_S(f)} = \dim_{\mathbb{K}} \text{Image}({}_{i+1}\bar{t}f) + {}_i\gamma(f).$$

Therefore, we have the following:

$$\dim_{\mathbb{K}} \frac{f^*m_0^{i+1}\theta_S(f)}{T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f) + f^*m_0^{i+2}\theta_S(f)} = (p-n) \cdot {}_{i+1}\delta(f) + {}_{i+1}\gamma(f) - {}_i\gamma(f).$$

Hence, we have the following:

$$\begin{aligned} & \dim_{\mathbb{K}} \ker({}_{i+1}\bar{\omega}f) \\ &= \dim_{\mathbb{K}} \frac{m_0^{i+1}\theta_0(p)}{m_0^{i+2}\theta_0(p)} - \dim_{\mathbb{K}} \frac{f^*m_0^{i+1}\theta_S(f)}{T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f) + f^*m_0^{i+2}\theta_S(f)} \\ &= p \cdot \binom{p+i}{i+1} - ((p-n) \cdot {}_{i+1}\delta(f) + {}_{i+1}\gamma(f) - {}_i\gamma(f)). \end{aligned}$$

Q.E.D.

Thirdly, Proposition 3 is proved.

Proof of Proposition 3

By Lemma 1.1, it suffices to show that for any i and any finitely determined multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) of corank at most one satisfying that ${}_i\bar{\omega}f$ is surjective, ${}_{i+1}\bar{\omega}f$ is not injective. By Lemma 1.1, ${}_{i+1}\bar{\omega}f$ is not injective if ${}_i\bar{\omega}f$ is not injective. Thus, we may assume that ${}_i\bar{\omega}f$ is bijective.

We first prove Proposition 3 in the case $i = 0$. Since we have assumed that ${}_0\overline{\omega}f$ is bijective, the following holds (see [17] or [28]):

$$p \cdot \binom{p-1}{0} = \dim_{\mathbb{K}} \frac{\theta_0(p)}{m_0\theta_0(p)} = (p-n) \cdot {}_0\delta(f) + {}_0\gamma(f).$$

Note that the above equality can not be obtained by Proposition 5. Note further that at least one of $p-n > 0$ or ${}_0\gamma(f) > 0$ holds by this equality. We have the following:

$$\begin{aligned} p \cdot \binom{p}{1} &= p^2 \cdot \binom{p-1}{0} \\ &= p \cdot ((p-n) \cdot {}_0\delta(f) + {}_0\gamma(f)) \\ &= \frac{p}{n} \cdot ((p-n) \cdot {}_1\delta(f) + {}_1\gamma(f)) \quad (\text{by 2 of Proposition 5}) \\ &\geq (p-n) \cdot {}_1\delta(f) + {}_1\gamma(f) \quad (\text{by } n \leq p) \\ &\geq (p-n) \cdot {}_1\delta(f) + {}_1\gamma(f) - {}_0\gamma(f) \quad (\text{by } {}_0\gamma(f) \geq 0). \end{aligned}$$

Since we have confirmed that at least one of $p-n > 0$ or ${}_0\gamma(f) > 0$ holds, we have the following sharp inequality:

$$p \cdot \binom{p}{1} > (p-n) \cdot {}_1\delta(f) + {}_1\gamma(f) - {}_0\gamma(f).$$

Hence ${}_1\overline{\omega}f$ is not injective by Lemma 1.1 and Proposition 5.

Next we prove Proposition 3 in the case $i \geq 1$. Since we have assumed that ${}_i\overline{\omega}f$ is bijective, we have the following equality by Proposition 5:

$$\begin{aligned} &p \cdot \binom{p+i-1}{i} \\ &= (p-n) \cdot {}_i\delta(f) + {}_i\gamma(f) - {}_{i-1}\gamma(f) \\ &= (p-n) \cdot {}_i\delta(f) + \left(1 - \frac{i}{n+i-1}\right) \cdot {}_i\gamma(f) \quad (\text{by 2 of Proposition 6}) \\ &= (p-n) \cdot {}_i\delta(f) + \frac{n-1}{n+i-1} \cdot {}_i\gamma(f). \end{aligned}$$

Note that at least one of $p - n > 0$ or $(n - 1) \cdot {}_i\gamma(f) > 0$ holds by this equality. We have the following:

$$\begin{aligned}
& p \cdot \binom{p+i}{i+1} \\
&= \frac{p+i}{i+1} \cdot p \cdot \binom{p+i-1}{i} \\
&= \frac{p+i}{i+1} \cdot \left((p-n) \cdot {}_i\delta(f) + \frac{n-1}{n+i-1} \cdot {}_i\gamma(f) \right) \\
&= \frac{p+i}{i+1} \cdot \left(\frac{i+1}{n+i} \cdot (p-n) \cdot {}_{i+1}\delta(f) + \frac{n-1}{n+i-1} \cdot \frac{i+1}{n+i} \cdot {}_{i+1}\gamma(f) \right) \\
&\quad \text{(by 2 of Proposition 5)} \\
&= \frac{p+i}{n+i} \cdot (p-n) \cdot {}_{i+1}\delta(f) + \frac{p+i}{n+i-1} \cdot \frac{n-1}{n+i} \cdot {}_{i+1}\gamma(f) \\
&\geq (p-n) \cdot {}_{i+1}\delta(f) + \frac{p+i}{n+i-1} \cdot \frac{n-1}{n+i} \cdot {}_{i+1}\gamma(f) \quad (\text{by } n \leq p) \\
&\geq (p-n) \cdot {}_{i+1}\delta(f) + \frac{n-1}{n+i} \cdot {}_{i+1}\gamma(f) \quad (\text{by } n \leq p \text{ and } (n-1)_{i+1}\gamma(f) \geq 0) \\
&= (p-n) \cdot {}_{i+1}\delta(f) + {}_{i+1}\gamma(f) - {}_i\gamma(f) \quad (\text{by 2 of Proposition 6}).
\end{aligned}$$

Since we have confirmed that at least one of $p - n > 0$ or $(n - 1) \cdot {}_i\gamma(f) > 0$ holds and ${}_{i+1}\gamma(f) = \frac{n+i}{i+1} \cdot {}_i\gamma(f)$ by the assertion 2 of Proposition 6, we have the following sharp inequality:

$$p \cdot \binom{p+i}{i+1} > (p-n) \cdot {}_{i+1}\delta(f) + {}_{i+1}\gamma(f) - {}_i\gamma(f).$$

Hence, ${}_{i+1}\overline{\omega}f$ is not injective by Lemma 1.1 and Proposition 5.

Q.E.D.

Finally, Proposition 4 is proved.

Proof of Proposition 4

Proof of the assertion (1) of Proposition 4 is as follows. Recall that ${}_0\overline{\omega}f$ is nothing but Mather's $\overline{\omega}f$ defined in [15]. The assertion (a) has been already shown by Mather (see p.228 of [15]). Since $\overline{\omega}f : \theta_0(p)/m_0\theta_0(p) \rightarrow \theta_S(f)/TK_e(f)$ is defined by $\overline{\omega}f([\eta]) = [\eta \circ f]$, by definition, the injectivity of $\overline{\omega}f$ is equivalent to assert that $ev_0(\eta) = \eta(0) = 0$ for any $\eta \in Lift(f)$.

Proof of the assertion (2) of Proposition 4 is as follows. Recall that ${}_1\overline{\omega}f$ is the following mapping defined by ${}_1\overline{\omega}f([\eta]) = [\eta \circ f]$:

$${}_1\overline{\omega}f : \frac{m_0\theta_0(p)}{m_0^2\theta_0(p)} \rightarrow \frac{f^*m_0\theta_S(f)}{T\mathcal{R}_e(f) \cap f^*m_0\theta_S(f) + f^*m_0^2\theta_S(f)}.$$

As we have already confirmed in Section 1, by the preparation theorem, we have that ${}_1\overline{\omega}f$ is surjective if and only if $f^*m_0\theta_S(f) \subset T\mathcal{A}_e(f)$. Since f is finitely determined, we can conclude that $f^*m_0\theta_S(f) \subset T\mathcal{A}_e(f)$ if and only if $f^*m_0\theta_S(f) \subset T\mathcal{A}(f)$.

For (b) of (2), it is easily seen that injectivity of ${}_1\overline{\omega}f$ is equivalent to assert that η has no constant terms and no linear terms for any $\eta \in Lift(f)$. These equivalent conditions imply that f is not \mathcal{A} -equivalent to a quasi-homogeneous multigerm. By [5], this implies that $\dim_{\mathbb{K}} \theta_S(f)/T\mathcal{A}_e(f) > 1$.

Q.E.D.

3. EXAMPLES OF THEOREM 1

Example 3.1. Let $\varphi : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ be the map-germ given by $\varphi(x_1, \dots, x_{n-1}, y) = (x_1, \dots, x_{n-1}, y^{n+1} + \sum_{i=1}^{n-1} x_i y^i)$. Then, it is known that f is an isolated stable mono-germ by [20] or [16]. Thus, by Proposition 4, ${}_0\overline{\omega}\varphi$ is bijective. Therefore, it follows that $i_1(\varphi) = i_2(\varphi) = 0$. By Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(\varphi)$ can be calculated as follows:

$$\begin{aligned} & n \cdot \binom{n}{1} - ((n-n) \cdot {}_1\delta(\varphi) + {}_1\gamma(\varphi) - {}_0\gamma(\varphi)) \\ &= n^2 - ((n-n) \cdot n \cdot (n+1) + n \cdot (n+1-1) - (n+1-1)) \\ &= n. \end{aligned}$$

It has been verified in [1] that the minimal number of generators for $Lift(\varphi)$ is exactly n in the complex case.

Example 3.2. Let $\varphi_k : (\mathbb{K}^{2k-2}, 0) \rightarrow (\mathbb{K}^{2k-1}, 0)$ be given by

$$\begin{aligned} & \varphi_k(u_1, \dots, u_{k-2}, v_1, \dots, v_{k-1}, y) \\ &= \left(u_1, \dots, u_{k-2}, v_1, \dots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i \right). \end{aligned}$$

Then, it is known that f is an isolated stable mono-germ by [20] or [16]. Thus, by Proposition 4, ${}_0\overline{\omega}\varphi_k$ is bijective. Therefore, it follows that $i_1(\varphi_k) = i_2(\varphi_k) = 0$. By Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(\varphi_k)$ can be calculated as follows:

$$\begin{aligned} & (2k-1) \cdot \binom{2k-1}{1} - (((2k-1) - (2k-2)) \cdot {}_1\delta(\varphi_k) + {}_1\gamma(\varphi_k) - {}_0\gamma(\varphi_k)) \\ &= (2k-1)^2 - (((2k-1) - (2k-2)) \cdot (2k-2) \cdot k \\ & \quad + (2k-2) \cdot (k-1) - (k-1)) \\ &= 3k-2. \end{aligned}$$

It has been verified in [9] that the minimal number of generators for $Lift(\varphi_k)$ is exactly $3k-2$ in the complex case and in the case a set of generators has been obtained in [11] (see also [4]). In Subsection 6.1, a set of linear parts of generators for $Lift(\varphi_k)$ is obtained by our method for any $k \geq 2$.

Example 3.3. Let $\psi_n : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^{2n-1}, 0)$ be given by

$$\psi_n(v_1, \dots, v_{n-1}, y) = (v_1, \dots, v_{n-1}, y^2, v_1 y, \dots, v_{n-1} y).$$

Then, it is known that f is an isolated stable mono-germ by [29] or [30] or [16]. Thus, by Proposition 4, ${}_0\overline{\omega}\psi_n$ is bijective. Therefore, it follows that $i_1(\psi_n) = i_2(\psi_n) = 0$. By Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(\psi_n)$ can be calculated as follows:

$$\begin{aligned} & (2n-1) \cdot \binom{2n-1}{1} - (((2n-1) - n) \cdot {}_1\delta(\varphi) + {}_1\gamma(\varphi) - {}_0\gamma(\varphi)) \\ &= (2n-1)^2 - ((n-1) \cdot n \cdot 2 + n \cdot (2-1) - (2-1)) \\ &= 2n^2 - 3n + 2. \end{aligned}$$

In the case that $n = 2$, ψ_2 equals φ_2 of Example 3.2. Thus, in this case, It has been verified in [4] and [9] that the minimal number of generators for $Lift(\psi_n)$ is

exactly 4 in the complex case and a set of generators has been obtained in [4] and [11]. In Subsection 6.2, a set of generators for $Lift(\psi_n)$ is obtained by our method for any $n \geq 2$.

Example 3.4. Examples 3.1, 3.2 and 3.3 can be generalized as follows. Let $f : (\mathbb{K}, 0) \rightarrow (\mathbb{K}^p, 0)$ ($p \geq 2$) be an analytic map-germ such that $2 \leq \delta(f) < \infty$ and let $F : (\mathbb{K} \times \mathbb{K}^c, 0) \rightarrow (\mathbb{K}^p \times \mathbb{K}^c, 0)$ be a \mathcal{K} -miniversal unfolding of f , where \mathcal{K} -miniversal unfolding of f is a map-germ given by (5.8) of [15] with $c = r$. Then, by [15] or [16], F is an isolated stable mono-germ. Thus, by Proposition 4, ${}_0\overline{\omega}F$ is bijective. Note that $c = p\delta(f) - 1 - p$ by theorem 4.5.1 of [28]. By Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(F)$ can be calculated as follows:

$$\begin{aligned} & (p+c) \cdot \binom{p+c}{1} - (((p+c) - (1+c)) \cdot {}_1\delta(F) + {}_1\gamma(F) - {}_0\gamma(F)) \\ &= (p+c)^2 - ((p-1) \cdot (1+c) \cdot \delta(f) + c \cdot (\delta(f) - 1)) \\ &= p^2 \cdot \delta(f) - p \cdot \delta(f) + \delta(f) - p. \end{aligned}$$

By Mather's classification theorem (theorem A of [15]), proposition (1.6) of [15], Mather's normal form theorem for a stable map-germ (theorem (5.10) of [15]), the fact that the sharp inequality $p^2\delta(f) - p\delta(f) + \delta(f) - p > p + c$ holds (since $p, \delta(f) \geq 2$) and the fact that the module of liftable vector fields over an immersive stable multigerms is a free module if and only if $p = n + 1$, we have the following:

Proposition 7. *Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n < p$) be a stable multigerms of corank at most one. Then, $Lift(f)$ is a free module if and only if the properties $p = n + 1$ and $\delta(f) = |S|$ are satisfied.*

Example 3.5. Let $f : (\mathbb{K}^2, S) \rightarrow (\mathbb{K}^2, 0)$ be given by $(x, y) \mapsto (x, y^2)$, $(x, y) \mapsto (x^2, y)$. Then, it is known that f is an isolated multigerms by [31] or [16]. Thus, by Proposition 4, ${}_0\overline{\omega}f$ is bijective. Therefore, it follows that $i_1(f) = i_2(f) = 0$. By Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(f)$ is the following:

$$\begin{aligned} & \dim_{\mathbb{K}} \frac{m_0\theta_0(2)}{m_0^2\theta_0(2)} - ((2-2){}_1\delta(f) + {}_1\gamma(f) - {}_0\gamma(f)) \\ &= 2^2 - ((2-2) \times 2 \times 4 + 2 \times (4-2) - (4-2)) \\ &= 2. \end{aligned}$$

In this case we can construct easily a basis of $Lift(f)$ consisting of 2 vector fields (see Subsection 6.4).

Example 3.6. Let $f : (\mathbb{K}, S) \rightarrow (\mathbb{K}^2, 0)$ be any one of the following three.

- (1) $x \mapsto (x^4, x^5 + x^7)$ (taken from [3]).
- (2) $x \mapsto (x^2, x^3), x \mapsto (x^3, x^2)$ (taken from [12]).
- (3) $x \mapsto (x, 0), x \mapsto (0, x), x \mapsto (x^2, x^3 + x^4)$ (taken from [12]).

It has been shown in [3] or [12] that $TK(f) = T\mathcal{A}(f)$ is satisfied. Thus, by Proposition 4, ${}_1\overline{\omega}f$ is surjective. We can confirm easily that the following equality holds.

$$2 \cdot \binom{2}{1} = (2-1) \cdot {}_1\delta(f) + {}_1\gamma(f) - {}_0\gamma(f).$$

Thus, ${}_1\overline{\omega}f$ is injective by Proposition 5. Therefore, it follows that $i_1(f) = i_2(f) = 1$. By Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(f)$ can be calculated as follows:

$$\begin{aligned} & 2 \cdot \binom{3}{2} - ((2-1) \cdot {}_2\delta(f) + {}_2\gamma(f) - {}_1\gamma(f)) \\ = & 2 \cdot 3 - ((2-1) \cdot 1 \cdot 4 + (4 - |S|) - (4 - |S|)) \\ = & 2. \end{aligned}$$

In the case $\mathbb{K} = \mathbb{C}$, it has been known that any plane algebraic curve is a free divisor by [27]. Thus, by combining [6] and [27], it has been known that the minimal number of generators for $Lift(f)$ is 2 in the complex case.

Example 3.7. Let $f : (\mathbb{K}^2, 0) \rightarrow (\mathbb{K}^2, 0)$ be given by $f(x, y) = (x, xy + y^5 \pm y^7)$ (taken from [24]). It has been shown in [24] that $TK(f) = T\mathcal{A}(f)$ is satisfied. Thus, by Proposition 4, ${}_1\overline{\omega}f$ is surjective. It is easily seen that the following equality holds.

$$2 \cdot \binom{2}{1} = (2-2) \cdot {}_1\delta(f) + {}_1\gamma(f) - {}_0\gamma(f).$$

Thus, ${}_1\overline{\omega}f$ is injective by Proposition 5. Therefore, it follows that $i_1(f) = i_2(f) = 1$. By Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(f)$ can be calculated as follows.

$$\begin{aligned} & 2 \cdot \binom{3}{2} - ((2-2) \cdot {}_2\delta(f) + {}_2\gamma(f) - {}_1\gamma(f)) \\ = & 2 \cdot 3 - ((2-2) \cdot 3 \cdot 5 + 3 \cdot (5-1) - 2 \cdot (5-1)) \\ = & 2. \end{aligned}$$

As same as Example 3.6, it has been known that the minimal number of generators for $Lift(f)$ is 2 in the complex case.

Example 3.8. Let $f : (\mathbb{K}^4, 0) \rightarrow (\mathbb{K}^5, 0)$ be given by the following:

$$f(x_1, x_2, x_3, y) = (x_1, x_2, x_3, y^4 + x_1y, y^6 + y^7 + x_2y + x_3y^2).$$

This example is taken from [26] where the property $TK(f) = T\mathcal{A}(f)$ has been shown. Thus, by Proposition 4, ${}_1\overline{\omega}f$ is surjective. It is easily seen that the following equality holds.

$$5 \cdot \binom{5}{1} = (5-4) \cdot {}_1\delta(f) + {}_1\gamma(f) - {}_0\gamma(f).$$

Thus, ${}_1\overline{\omega}f$ is injective by Proposition 5. Therefore, it follows that $i_1(f) = i_2(f) = 1$. By Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(f)$ can be calculated as follows:

$$\begin{aligned} & 5 \cdot \binom{6}{2} - ((5-4) \cdot {}_2\delta(f) + {}_2\gamma(f) - {}_1\gamma(f)) \\ = & 5 \cdot 15 - ((5-4) \cdot 10 \cdot 4 + 10 \cdot (4-1) - 4 \cdot (4-1)) \\ = & 17. \end{aligned}$$

Example 3.9. Let $c : \mathbb{K} \rightarrow \mathbb{K}^2$ be the map defined by $c(x) = (x^2, x^3)$. For any real number θ , we let $R_\theta : \mathbb{K}^2 \rightarrow \mathbb{K}^2$ be the rotation of \mathbb{K}^2 about the origin with

respect to the angle θ :

$$R_\theta \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

For any $i \in \mathbb{N}$, let $\theta_0, \dots, \theta_i$ be real numbers such that $0 \leq \theta_j < 2\pi$ ($0 \leq j \leq i$) and $0 \neq |\theta_j - \theta_k| \neq \pi$ ($j \neq k$). Set $S = \{s_0, \dots, s_i\}$ ($s_j \neq s_k$ if $j \neq k$) and define $c_{\theta_j} : (\mathbb{K}, s_j) \rightarrow (\mathbb{K}^2, 0)$ as $c_{\theta_j}(x) = R_{\theta_j} \circ c(x_j)$, where $x_j = x - s_j$. A multigerm $\{c_{\theta_0}, \dots, c_{\theta_i}\} : (\mathbb{K}, S) \rightarrow (\mathbb{K}^2, 0)$, which is called a *multicusp*, is denoted by $c_{(\theta_0, \dots, \theta_i)}$.

In [18], it has been shown that $i_1(c_{(\theta_0, \dots, \theta_i)}) = i_2(c_{(\theta_0, \dots, \theta_i)}) = i$ for any $i \in \mathbb{N}$. Thus, by Theorem 1, Lemma 1.1 and Propositions 5, 6, the minimal number of generators for $Lift(c_{(\theta_0, \dots, \theta_i)})$ can be calculated as follows.

$$\begin{aligned} & 2 \cdot \binom{2+i}{i+1} - ((2-1) \cdot i_{+1} \delta(c_{(\theta_0, \dots, \theta_i)}) + i_{+1} \gamma(c_{(\theta_0, \dots, \theta_i)}) - i \gamma(c_{(\theta_0, \dots, \theta_i)})) \\ &= 2 \cdot (2+i) - ((2-1) \cdot 1 \cdot \delta(c_{(\theta_0, \dots, \theta_i)}) + 1 \cdot \gamma(c_{(\theta_0, \dots, \theta_i)}) - 1 \cdot \gamma(c_{(\theta_0, \dots, \theta_i)})) \\ &= 2. \end{aligned}$$

As same as Example 3.6, it has been already known that the minimal number of generators for $Lift(c_{(\theta_0, \dots, \theta_i)})$ is 2 in the complex case.

4. PROOF OF THEOREM 1

Since $i\bar{\omega}f$ is surjective, by Lemma 1.1 we have that $j\bar{\omega}f$ is surjective for any $j > i$. Since $i\bar{\omega}f$ is injective, any $\eta \in \theta_0(p)$ such that $\omega f(\eta) \in T\mathcal{R}_e(f)$ is contained in $m_0^{i+1}\theta_0(p)$. Set $\rho(f) = \dim_{\mathbb{K}} \ker(i_{+1}\bar{\omega}f)$. Then, since $i\bar{\omega}f$ is bijective, $\rho(f)$ must be positive by Corollary 1. Let $\{\eta_1 + m_0^{i+2}\theta_0(p), \dots, \eta_{\rho(f)} + m_0^{i+2}\theta_0(p)\}$ be a basis of $\ker(i_{+1}\bar{\omega}f)$. Then, we have that

$$\eta_j \circ f \in T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f) + f^*m_0^{i+2}\theta_S(f) \quad (1 \leq j \leq \rho(f)).$$

Since $i_{+2}\bar{\omega}f$ is surjective, we have the following:

$$\begin{aligned} & T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f) + f^*m_0^{i+2}\theta_S(f) \\ &= T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f) + T\mathcal{R}_e(f) \cap f^*m_0^{i+2}\theta_S(f) + \omega f(m_0^{i+2}\theta_0(p)). \end{aligned}$$

Thus, for any j ($1 \leq j \leq \rho(f)$) there exists $\tilde{\eta}_j \in m_0^{i+2}\theta_0(p)$ such that $(\eta_j + \tilde{\eta}_j) \circ f \in T\mathcal{R}_e(f) \cap T\mathcal{L}_e(f)$. Let A be the C_0 -module generated by $\eta_j + \tilde{\eta}_j$ ($1 \leq j \leq \rho(f)$).

Let $\hat{\omega}f : \theta_0(p) \rightarrow \frac{\theta_S(f)}{T\mathcal{R}_e(f)}$ be given by $\hat{\omega}f(\eta) = \omega f(\eta) + T\mathcal{R}_e(f)$. Then, $\ker(\hat{\omega}f)$ is the set of vector fields liftable over f . In order to show that $\ker(\hat{\omega}f) = A$, we consider the following commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \ker(b_3) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ m_0\ker(\hat{\omega}f) & \xrightarrow{a_2} & m_0^{i+2}\theta_0(p) & \xrightarrow{c_2} & \frac{f^*m_0^{i+2}\theta_S(f)}{f^*m_0(T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f))} & \longrightarrow & 0 \\ \downarrow b_1 & & \downarrow b_2 & & \downarrow b_3 & & \\ 0 \longrightarrow & \ker(\hat{\omega}f) & \xrightarrow{a_1} & m_0^{i+1}\theta_0(p) & \xrightarrow{c_1} & \frac{f^*m_0^{i+1}\theta_S(f)}{T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f)} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{coker}(b_1) & \xrightarrow{d_1} & \text{coker}(b_2) & \xrightarrow{d_2} & \text{coker}(b_3) & & \end{array}$$

Here, a_j ($j = 1, 2$), b_j ($j = 1, 2$) are inclusions, b_3 is defined by $b_3([\eta]_{i+2}) = [\eta]_{i+1}$ and c_j ($j = 1, 2$) are defined by $c_j(\eta) = [\omega f(\eta)]_{i+j}$, where $[\eta]_{i+1} = \eta + T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f)$ and $[\eta]_{i+2} = \eta + f^*m_0(T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f))$.

Lemma 4.1.

$$f^*m_0(T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f)) = T\mathcal{R}_e(f) \cap f^*m_0^{i+2}\theta_S(f).$$

Proof of Lemma 4.1. It is clear that $f^*m_0(T\mathcal{R}_e(f) \cap f^*m_0^{i+1}\theta_S(f)) \subset T\mathcal{R}_e(f) \cap f^*m_0^{i+2}\theta_S(f)$. Thus, in the following we concentrate on showing its converse. Let η be an element of $T\mathcal{R}_e(f) \cap f^*m_0^{i+2}\theta_S(f)$. Let f_j be a branch of f , namely, $f_j = f|_{(\mathbb{K}^n, s_j)}$ ($1 \leq j \leq |S|$). Then, we have that $\eta \in T\mathcal{R}_e(f_j) \cap f_j^*m_0^{i+2}\theta_{s_j}(f_j)$ for any j ($1 \leq j \leq |S|$). Thus, there exists $\xi_j \in \theta_{s_j}(n)$ such that $tf_j(\xi_j) = \eta$.

Since f is of corank at most one, for any j ($1 \leq j \leq |S|$) there exist germs of diffeomorphism $h_j : (\mathbb{K}^n, s_j) \rightarrow (\mathbb{K}^n, s_j)$ and $H_j : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$ such that $H_j \circ f_j \circ h_j^{-1}$ has the following form:

$$\begin{aligned} & H_j \circ f_j \circ h_j^{-1}(x, y) \\ &= (x, y^{\delta(f_j)} + f_{j,n}(x, y), f_{j,n+1}(x, y), \dots, f_{j,p}(x, y)). \end{aligned}$$

Here, x stands for (x_1, \dots, x_{n-1}) and x_1, \dots, x_{n-1}, y are local coordinates of the coordinate system (U_j, h_j) at s_j and $f_{j,q}$ satisfies $f_{j,q}(0, \dots, 0, y) = o(y^{\delta(f_j)})$ for any q ($n \leq q \leq p$). Set

$$\xi_j = \sum_{m=1}^{n-1} \xi_{j,m} \frac{\partial}{\partial x_m} + \xi_{j,n} \frac{\partial}{\partial y} \text{ and } \eta = \sum_{q=1}^p \eta_q \frac{\partial}{\partial X_q}.$$

Then, by the above form of $H_j \circ f_j \circ h_j^{-1}$ and the equality $tf_j(\xi_j) = \eta$, the following hold:

$$\begin{aligned} (1) \quad & \xi_{j,m}(x_1, \dots, x_{n-1}, y) = \eta_m(x_1, \dots, x_{n-1}, y) \quad (1 \leq m \leq n-1) \\ (2) \quad & \lambda(x_1, \dots, x_{n-1}, y) \xi_{j,n}(x_1, \dots, x_{n-1}, y) = \mu(x_1, \dots, x_{n-1}, y), \end{aligned}$$

where $\lambda = \delta(f_j)y^{\delta(f_j)-1} + \frac{\partial f_{j,n}}{\partial y}$ and $\mu = \eta_n - \sum_{m=1}^{n-1} \xi_{j,m} \frac{\partial f_{j,n}}{\partial x_m}$. Since $\eta_q \in f^*m_0^{i+2}C_{s_j}$ for any q ($1 \leq q \leq p$), by (4.1) we have that $\xi_{j,m} \in f^*m_0^{i+2}C_{s_j}$ for any m ($1 \leq m \leq n-1$).

Since $f_{j,q}(0, \dots, 0, y) = o(y^{\delta(f_j)})$ for any q ($n \leq q \leq p$), we have the following properties:

$$\begin{aligned} (1) \quad & Q(f_j) = Q(x_1, \dots, x_{n-1}, y^{\delta(f_j)}) = Q(x_1, \dots, x_n, y^\lambda). \\ (2) \quad & [1], [y], \dots, [y^{\delta(f_j)-2}], [\lambda] \text{ constitute a basis of } Q(f_j). \end{aligned}$$

Thus, by the preparation theorem, C_{s_j} is generated by $1, y, \dots, y^{\delta(f_j)-2}, \lambda$ as C_0 -module via f_j . Therefore, for any positive integer r , $f_j^*m_0^r C_{s_j}$ is generated by

elements of the union of the following three sets U_r, V_r, W_r as C_0 -module via f_j .

$$\begin{aligned} U_r &= \left\{ x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} \lambda^{k_n} y^{k_n+\ell} \mid k_m \geq 0, \sum_{m=1}^{n-1} k_m = r - k_n < r, 0 \leq \ell \leq \delta(f_j) - 2 \right\}, \\ V_r &= \left\{ x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} \lambda^{k_n+1} y^{k_n} \mid k_m \geq 0, \sum_{m=1}^{n-1} k_m = r - k_n \right\}, \\ W_r &= \left\{ x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} y^\ell \mid k_m \geq 0, \sum_{m=1}^{n-2} k_m = r, 0 \leq \ell \leq \delta(f_j) - 2 \right\}. \end{aligned}$$

Then, by using these notations, for any m ($1 \leq m \leq n-1$) $\xi_{j,m}$ can be expressed as follows:

$$\xi_{j,m} = \sum_{u \in U_{i+2}} \varphi_{u,j,m} u + \sum_{v \in V_{i+2}} \varphi_{v,j,m} v + \sum_{w \in W_{i+2}} \varphi_{w,j,m} w,$$

where $\varphi_{u,j,m}, \varphi_{v,j,m}, \varphi_{w,j,m}$ are some elements of C_{s_j} .

Next, we investigate $\xi_{j,n}$. Since μ has the form $\mu = \eta_n - \sum_{m=1}^{n-1} \xi_{j,m} \frac{\partial f_{j,n}}{\partial x_m}$ and $\eta_n, \xi_{j,m}$ are contained in $f^* m_0^{i+2} C_{s_j}$, μ is contained in $f^* m_0^{i+2} C_{s_j}$. On the other hand, λ must divide μ by (4.2). Thus, μ is generated by elements of $U_{i+2} \cup V_{i+2}$. Hence, $\xi_{j,n} = \frac{\mu}{\lambda}$ can be expressed as follows:

$$\xi_{j,n} = \sum_{u \in U_{i+2}} \varphi_{u,j,n} \frac{u}{\lambda} + \sum_{v \in V_{i+2}} \varphi_{v,j,n} \frac{v}{\lambda},$$

where $\varphi_{u,j,n}, \varphi_{v,j,n}$ are some elements of C_{s_j} . Since $\frac{u}{\lambda} \in U_{i+1} \cup V_{i+1} \cup W_{i+1}$ for any $u \in U_{i+2}$ and $\frac{v}{\lambda} \in U_{i+1}$ (resp., $\frac{v}{\lambda} \in V_{i+1}$) if $\delta(f_j) \geq 2$ (resp., $\delta(f_j) = 1$) for any $v \in V_{i+2}$, $\xi_{j,n}$ is belonging to $f_j^* m_0^{i+1} C_{s_j}$.

Since $f_j^* m_0^{i+2} C_{s_j} \subset f_j^* m_0^{i+1} C_{s_j}$, for any j ($1 \leq j \leq |S|$) and any m ($1 \leq m \leq n-1$), we have the following:

$$\begin{aligned} & tf_j \left(\xi_{j,m} \frac{\partial}{\partial x_m} \right) \\ &= tf_j \left(\left(\sum_{u \in U_{i+1}} \tilde{\varphi}_{u,j,m} u + \sum_{v \in V_{i+1}} \tilde{\varphi}_{v,j,m} v + \sum_{w \in W_{i+1}} \tilde{\varphi}_{w,j,m} w \right) \frac{\partial}{\partial x_m} \right) \\ &= \sum_{u \in U_{i+1}} u \left(tf_j \left(\tilde{\varphi}_{u,j,m} \frac{\partial}{\partial x_m} \right) \right) + \sum_{v \in V_{i+1}} v \left(tf_j \left(\tilde{\varphi}_{v,j,m} \frac{\partial}{\partial x_m} \right) \right) \\ &\quad + \sum_{w \in W_{i+1}} w \left(tf_j \left(\tilde{\varphi}_{w,j,m} \frac{\partial}{\partial x_m} \right) \right), \end{aligned}$$

where $\tilde{\varphi}_{u,j,m}, \tilde{\varphi}_{v,j,m}, \tilde{\varphi}_{w,j,m}$ are some elements of C_{s_j} . Moreover, for any j ($1 \leq j \leq |S|$) we have the following:

$$\begin{aligned} & tf_j \left(\xi_{j,n} \frac{\partial}{\partial y} \right) \\ &= tf_j \left(\left(\sum_{u \in U_{i+1}} \psi_{u,j,n} u + \sum_{v \in V_{i+1}} \psi_{v,j,n} v + \sum_{w \in W_{i+1}} \psi_{w,j,n} w \right) \frac{\partial}{\partial y} \right) \\ &= \sum_{u \in U_{i+1}} u \left(tf_j \left(\psi_{u,j,n} \frac{\partial}{\partial y} \right) \right) + \sum_{v \in V_{i+1}} v \left(tf_j \left(\psi_{v,j,n} \frac{\partial}{\partial y} \right) \right) \\ &\quad + \sum_{w \in W_{i+1}} w \left(tf_j \left(\psi_{w,j,n} \frac{\partial}{\partial y} \right) \right), \end{aligned}$$

where $\psi_{u,j,n}, \psi_{v,j,n}, \psi_{w,j,n}$ are elements of C_{s_j} . Since the union $U_{i+1} \cup V_{i+1} \cup W_{i+1}$ is a finite set, we have the following:

$$\eta = tf_j \left(\sum_{m=1}^{n-1} \xi_{j,m} \frac{\partial}{\partial x_m} + \xi_{j,n} \frac{\partial}{\partial y} \right) \in f_j^* m_0^{i+1} (T\mathcal{R}_e(f_j) \cap f_j^* m_0 \theta_{s_j}(f_j)).$$

Notice that $i+1 \geq 1$ and f_j is any branch of f . Hence, we have that

$$\eta \in f^* m_0 (T\mathcal{R}_e(f) \cap f^* m_0^{i+1} \theta_S(f)).$$

□

Lemma 4.1 implies that c_2 is surjective, thus even the second row sequence is exact. Lemma 4.1 implies also that b_3 is injective and thus $\ker(b_3) = 0$. Hence, by the snake lemma, we see that d_1 is injective. On the other hand, since there exists an isomorphism

$$\varphi : \frac{f^* m_0^{i+1} \theta_S(f)}{T\mathcal{R}_e(f) \cap f^* m_0^{i+1} \theta_S(f) + f^* m_0^{i+2} \theta_S(f)} \rightarrow \text{coker}(b_3)$$

such that $d_2 = \varphi \circ {}_{i+1}\bar{\omega}f$ we have that $\ker(d_2) = \ker(\varphi \circ {}_{i+1}\bar{\omega}f) = \ker({}_{i+1}\bar{\omega}f)$. Therefore, we have the following:

$$\dim_{\mathbb{K}} \frac{\ker(\hat{\omega}f)}{m_0 \ker(\hat{\omega}f)} = \dim_{\mathbb{K}} \ker({}_{i+1}\bar{\omega}f) = \rho(f) = \dim_{\mathbb{K}} \frac{A}{m_0 A}.$$

Moreover, A is a submodule of $\ker(\hat{\omega}f)$ and our category is the analytic category. Therefore, we have that $\ker(\hat{\omega}f) = A$. Q.E.D.

5. PROOF OF THEOREM 2

We first show the assertion (1) of Theorem 2. Let $F(x, \lambda) = (f_\lambda(x), \lambda)$ ($f_0 = f$) be a stable unfolding of f . Since η is an element of $Lift(F)$, by definition, there exists a vector field $\xi \in \theta_S(n+r)$ such that $dF \circ \xi = \eta \circ F$. Set $x = (x_1, \dots, x_n)$, $\lambda = (\lambda_1, \dots, \lambda_r)$, $f_\lambda = (f_{\lambda,1}, \dots, f_{\lambda,p})$ and $\xi(x, \lambda) = (\xi_1(x, \lambda), \dots, \xi_{n+r}(x, \lambda))$. Then, we

have the following:

$$\begin{pmatrix} \frac{\partial f_\lambda}{\partial x}(x, \lambda) & \frac{\partial f_\lambda}{\partial \lambda}(x, \lambda) \\ 0 & E_r \end{pmatrix} \begin{pmatrix} \xi_1(x, \lambda) \\ \vdots \\ \xi_n(x, \lambda) \\ \xi_{n+1}(x, \lambda) \\ \vdots \\ \xi_{n+r}(x, \lambda) \end{pmatrix} = \begin{pmatrix} \eta_1(F(x, \lambda)) \\ \vdots \\ \eta_p(F(x, \lambda)) \\ \eta_{p+1}(F(x, \lambda)) \\ \vdots \\ \eta_{p+r}(F(x, \lambda)) \end{pmatrix}.$$

Here, $\frac{\partial f_\lambda}{\partial x}(x, \lambda)$ is the $p \times n$ matrix whose (i, j) elements is $\frac{\partial f_{\lambda, i}}{\partial x_j}(x, \lambda)$, $\frac{\partial f_\lambda}{\partial \lambda}(x, \lambda)$ is the $p \times r$ matrix whose (i, k) elements is $\frac{\partial f_{\lambda, i}}{\partial \lambda_k}(x, \lambda)$, and E_r stands for the $r \times r$ unit matrix. In particular, we have the following two:

$$\left(\frac{\partial f_\lambda}{\partial x}(x, 0) \right) \begin{pmatrix} \xi_1(x, 0) \\ \vdots \\ \xi_n(x, 0) \end{pmatrix} + \left(\frac{\partial f_\lambda}{\partial \lambda}(x, 0) \right) \begin{pmatrix} \xi_{n+1}(x, 0) \\ \vdots \\ \xi_{n+r}(x, 0) \end{pmatrix} = \begin{pmatrix} \eta_1(F(x, 0)) \\ \vdots \\ \eta_p(F(x, 0)) \end{pmatrix}$$

and

$$(*) \quad \xi_{n+k}(x, 0) = \eta_{p+k}(F(x, 0)) = \eta_{p+k}(f(x), 0) = 0 \quad (1 \leq k \leq r).$$

The last equality of $(*)$ is obtained from the assumption that $\eta \in \text{Lift}(g)$ and the fact that $\text{Lift}(g)$ is generated by $\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_p}$ and $\Lambda_1 \frac{\partial}{\partial \Lambda_1}, \dots, \Lambda_r \frac{\partial}{\partial \Lambda_r}$. Thus, we have the following:

$$\left(\frac{\partial f_\lambda}{\partial x}(x, 0) \right) \begin{pmatrix} \xi_1(x, 0) \\ \vdots \\ \xi_n(x, 0) \end{pmatrix} = \begin{pmatrix} \eta_1(F(x, 0)) \\ \vdots \\ \eta_p(F(x, 0)) \end{pmatrix}.$$

Therefore, $\bar{\eta}$ is a liftable vector field of f .

We next show the assertion (2) of Theorem 2. Since $\bar{\eta} \in \text{Lift}(f)$, by definition, there exists a vector field $\bar{\xi} \in \theta_S(n)$ such that $d\bar{f}(\bar{\xi}) = \bar{\eta} \circ f$. Set $\eta(X, \Lambda) = (\bar{\eta}(X), 0)$ and $\xi(x, \lambda) = (\bar{\xi}(x), 0)$. Set also $\tilde{\eta} = \eta \circ F - dF(\xi) \in \theta_{S \times \{0\}}(F)$. It is not difficult to see that $\tilde{\eta}(x, 0) = (0, 0)$. Thus, by the preparation theorem, there must exist a vector field $\tilde{\eta}_1 \in \theta_{S \times \{0\}}(F)$ such that $\tilde{\eta} = \Lambda \tilde{\eta}_1$.

Since F is stable, there exist $\hat{\xi} \in \theta_{S \times \{0\}}(n+1)$ and $\hat{\eta} \in \theta_{(0,0)}(p+1)$ such that $\tilde{\eta}_1 = dF(\hat{\xi}) + \hat{\eta} \circ F$. We therefore have that

$$\eta \circ F - dF(\xi) = \tilde{\eta} = \Lambda \tilde{\eta}_1 = \lambda(dF(\hat{\xi}) + \hat{\eta} \circ F) = dF(\lambda \hat{\xi}) + (\Lambda \hat{\eta}) \circ F.$$

It is clear that $\eta - \Lambda \hat{\eta} \in \text{Lift}(F)$. Moreover, we have that the $(p+1)$ component of $\eta - \Lambda \hat{\eta}$ is $0 - (\Lambda \hat{\eta})_{p+1} = -\Lambda(\hat{\eta})_{p+1}$, which implies that $\eta - \Lambda \hat{\eta} \in \text{Lift}(g_1)$. Q.E.D.

6. HOW TO CONSTRUCT LIFTABLE VECTOR FIELDS

In principle, the proof of Theorem 1 provides how to construct generators for the module of liftable vector fields over a given finitely determined multigerm f satisfying the assumption of Theorem 1. In Subsections 6.1–6.5, we examine it by several examples. In Subsections 6.6 and 6.7, as an application of Theorem 2, we explain how to construct all liftable vector fields over a given analytic multigerm admitting a one-parameter stable unfolding by several examples.

6.1. *Lift*(φ_k) for $\varphi_k(u_1, \dots, u_{k-2}, v_1, \dots, v_{k-1}, y) = (u_1, \dots, u_{k-2}, v_1, \dots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i)$.

Since the purpose of this subsection is to examine that the proof of Theorem 1 in principle provides how to construct generators for the module of liftable vector fields, in order to avoid just long calculations, in this subsection we restrict ourselves to obtain only the linear parts of generators for the module of liftable vector fields over φ_k of Example 3.2. Note that the linear parts of generators themselves are already useful to obtain the best lower bound for the \mathcal{A}_e codimensions of some multiterms (see [22]).

Definition 6.1. Let η_1, η_2 be vector fields along φ_k (namely, $\eta_1, \eta_2 \in \theta_S(\varphi_k)$).

- (1) We denote $\eta_1 \equiv \eta_2 \pmod{T\mathcal{R}_e(\varphi_k)}$ if $\eta_1 - \eta_2 \in T\mathcal{R}_e(\varphi_k)$.
- (2) We denote $\eta_1 \equiv \eta_2 \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^* m_0^2 \theta_S(\varphi_k)}$ if $\eta_1 - \eta_2 \in T\mathcal{R}_e(\varphi_k) + \varphi_k^* m_0^2 \theta_S(\varphi_k)$.

Let $(U_1, \dots, U_{k-2}, V_1, \dots, V_{k-1}, W_1, W_2)$ be the standard coordinates of \mathbb{K}^{2k-1} . Along the proof of Theorem 1, we first look for clues of linear terms of liftable vector fields in $\frac{\varphi_k^* m_0 \theta_S(\varphi_k)}{\varphi_k^* m_0^2 \theta_S(\varphi_k)}$. From the form of the Jacobian matrix of φ_k and since the minimal number of generators is $(3k-2)$ by Example 3.2, we can guess that clues of linear terms of liftable vector fields are the following $(3k-2)$ vector fields along φ_k :

$$\begin{aligned} & (W_1 \circ \varphi_k) \frac{\partial}{\partial W_1}, (W_2 \circ \varphi_k) \frac{\partial}{\partial W_1}, (W_2 \circ \varphi_k) \frac{\partial}{\partial W_2}, \\ & y^i (W_1 \circ \varphi_k) \frac{\partial}{\partial W_1} \quad (1 \leq i \leq k-2), \\ & y^i (W_2 \circ \varphi_k) \frac{\partial}{\partial W_1} \quad (1 \leq i \leq k-2), \\ & y^i (W_2 \circ \varphi_k) \frac{\partial}{\partial W_2} \quad (1 \leq i \leq k-1). \end{aligned}$$

First we try to find a vector field $\eta \in m_0 \theta_0(2k-1)$ such that $W_1 \frac{\partial}{\partial W_1} \neq \eta$ and $(W_1 \circ \varphi_k) \frac{\partial}{\partial W_1} \equiv \eta \circ \varphi_k \pmod{T\mathcal{R}_e(\varphi_k)}$ because $W_1 \frac{\partial}{\partial W_1} - \eta$ must be a liftable vector field for such a η .

$$\begin{aligned} & (W_1 \circ \varphi_k) \frac{\partial}{\partial W_1} \\ &= \left(y^k + \sum_{i=1}^{k-2} u_i y^i \right) \frac{\partial}{\partial W_1} \\ &\equiv \left(-\frac{1}{k} \sum_{i=1}^{k-2} i u_i y^i + \sum_{i=1}^{k-2} u_i y^i \right) \frac{\partial}{\partial W_1} - \sum_{i=1}^{k-1} i v_i y^{i-1} \frac{\partial}{\partial W_2} \pmod{T\mathcal{R}_e(\varphi_k)} \\ &\equiv -\sum_{i=1}^{k-2} \frac{k-i}{k} (U_i \circ \varphi_k) \frac{\partial}{\partial U_i} + \sum_{i=2}^{k-1} i (V_i \circ \varphi_k) \frac{\partial}{\partial V_{i-1}} \\ &\quad + (V_1 \circ \varphi_k) \frac{\partial}{\partial W_2} \pmod{T\mathcal{R}_e(\varphi_k)}. \end{aligned}$$

Thus, it follows that the following is a liftable vector fields over φ_k where $\tilde{\eta}_1 = 0$.

$$\eta_1 + \tilde{\eta}_1 = \sum_{i=1}^{k-2} \frac{k-i}{k} U_i \frac{\partial}{\partial U_i} - \sum_{i=2}^{k-1} i V_i \frac{\partial}{\partial V_{i-1}} + W_1 \frac{\partial}{\partial W_1} - V_1 \frac{\partial}{\partial W_2}.$$

Secondly, we try to find a vector field $\eta \in m_0\theta_0(2k-1)$ such that $W_2 \frac{\partial}{\partial W_1} \neq \eta$ and $(W_2 \circ \varphi_k) \frac{\partial}{\partial W_1} \equiv \eta \circ \varphi_k \pmod{T\mathcal{R}_e(\varphi_k)} + \varphi_k^2 m_0^2 \theta(\varphi_k)$.

$$\begin{aligned} (W_2 \circ \varphi_k) \frac{\partial}{\partial W_1} &= \left(\sum_{i=1}^{k-1} v_i y^i \right) \frac{\partial}{\partial W_1} \\ &\equiv - \sum_{i=1}^{k-2} (V_i \circ \varphi_k) \frac{\partial}{\partial U_i} \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2 \theta(\varphi_k)}. \end{aligned}$$

Thus, it follows that there exists a liftable vector field over φ_k having the following form, where $\tilde{\eta}_2 \in m_0^2 \theta_0(p)$.

$$\eta_2 + \tilde{\eta}_2 = \sum_{i=1}^{k-2} V_i \frac{\partial}{\partial U_i} + W_2 \frac{\partial}{\partial W_1} + \text{higher terms.}$$

Thirdly, we try to find a vector field $\eta \in m_0\theta_0(2k-1)$ such that $W_2 \frac{\partial}{\partial W_2} \neq \eta$ and $(W_2 \circ \varphi_k) \frac{\partial}{\partial W_2} \equiv \xi \circ \varphi_k \pmod{T\mathcal{R}_e(\varphi_k)}$.

$$\begin{aligned} (W_2 \circ \varphi_k) \frac{\partial}{\partial W_2} &= \left(\sum_{i=1}^{k-1} v_i y^i \right) \frac{\partial}{\partial W_2} \\ &\equiv - \sum_{i=1}^{k-1} (V_i \circ \varphi_k) \frac{\partial}{\partial V_i} \pmod{T\mathcal{R}_e(\varphi_k)}. \end{aligned}$$

Thus, it follows that the following is a liftable vector field over φ_k where $\tilde{\eta}_3 = 0$.

$$\eta_3 + \tilde{\eta}_3 = \sum_{i=1}^{k-1} V_i \frac{\partial}{\partial V_i} + W_2 \frac{\partial}{\partial W_2}.$$

Fourthly, since $y^i (W_1 \circ \varphi_k) \frac{\partial}{\partial W_1} \equiv -(W_1 \circ \varphi_k) \frac{\partial}{\partial U_i} \pmod{T\mathcal{A}_e(\varphi_k)}$ for any i ($1 \leq i \leq k-2$), we try to find a vector field $\eta_i \in m_0\theta_0(2k-1)$ such that $-W_1 \frac{\partial}{\partial U_i} \neq \eta_i$ and $y^i (W_1 \circ \varphi_k) \frac{\partial}{\partial W_1} \equiv \eta_i \circ \varphi_k \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2 \theta(\varphi_k)}$.

$$\begin{aligned} &y^i (W_1 \circ \varphi_k) \frac{\partial}{\partial W_1} \\ &= y^i \left(y^k + \sum_{j=1}^{k-2} u_j y^j \right) \frac{\partial}{\partial W_1} \\ &\equiv \left(-\frac{1}{k} \sum_{j=1}^{k-2} u_j y^{i+j} + \sum_{j=1}^{k-2} u_j y^{i+j} \right) \frac{\partial}{\partial W_1} - \sum_{j=1}^{k-1} j v_j y^{i+j} \frac{\partial}{\partial W_2} \pmod{T\mathcal{R}_e(\varphi_k)}. \\ &\equiv - \sum_{j=1}^{k-2-i} \frac{(k-j)}{k} (U_j \circ \varphi_k) \frac{\partial}{\partial U_{i+j}} + \sum_{j=1}^{k-1-i} j (V_j \circ \varphi_k) \frac{\partial}{\partial V_{i+j}} \\ &\quad \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2 \theta(\varphi_k)}. \end{aligned}$$

Thus, it follows that for any i ($1 \leq i \leq k-2$) there exists a liftable vector field over φ_k having the following form where $\tilde{\eta}_{3+i} \in m_0^2\theta_0(p)$.

$$\begin{aligned} & \eta_{3+i} + \tilde{\eta}_{3+i} \\ = & W_1 \frac{\partial}{\partial U_i} - \sum_{j=1}^{k-2-i} \frac{(k-j)}{k} U_j \frac{\partial}{\partial U_{i+j}} + \sum_{j=1}^{k-1-i} j V_j \frac{\partial}{\partial V_{i+j}} + \text{higher terms.} \end{aligned}$$

Fifthly, since $y^i(W_2 \circ \varphi_k) \frac{\partial}{\partial W_1} \equiv -(W_2 \circ \varphi_k) \frac{\partial}{\partial U_i} \pmod{T\mathcal{A}_e(\varphi_k)}$ for any i ($1 \leq i \leq k-2$), we try to find a vector field $\eta_i \in m_0\theta_0(2k-1)$ such that $-W_2 \frac{\partial}{\partial U_i} \neq \eta_i$ and $y^i(W_2 \circ \varphi_k) \frac{\partial}{\partial W_1} \equiv \eta_i \circ \varphi_k \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2\theta(\varphi_k)}$.

$$\begin{aligned} & y^i(W_2 \circ \varphi_k) \frac{\partial}{\partial W_1} \\ = & \left(\sum_{j=1}^{k-1} v_j y^{i+j} \right) \frac{\partial}{\partial W_1} \\ \equiv & \begin{cases} -\sum_{j=1}^{k-2-i} (V_j \circ \varphi_k) \frac{\partial}{\partial U_{i+j}} \\ \quad \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2\theta(\varphi_k)} & (1 \leq i \leq k-3), \\ 0 & \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2\theta(\varphi_k)} & (i = k-2). \end{cases} \end{aligned}$$

Thus, it follows that for any i ($1 \leq i \leq k-3$) there exists a liftable vector field over φ_k having the following form where $\tilde{\eta}_{k+1+i} \in m_0^2\theta_0(p)$,

$$\eta_{k+1+i} + \tilde{\eta}_{k+1+i} = W_2 \frac{\partial}{\partial U_i} - \sum_{j=1}^{k-2-i} V_j \frac{\partial}{\partial U_{i+j}} + \text{higher terms;}$$

and there exists a liftable vector field over φ_k having the following form, where $\tilde{\eta}_{2k-1} \in m_0^2\theta_0(p)$.

$$\eta_{2k-1} + \tilde{\eta}_{2k-1} = W_2 \frac{\partial}{\partial U_{k-2}} + \text{higher terms.}$$

Sixthly, since $y^i(W_2 \circ \varphi_k) \frac{\partial}{\partial W_2} \equiv -(W_2 \circ \varphi_k) \frac{\partial}{\partial V_i} \pmod{T\mathcal{A}_e(\varphi_k)}$ for any i ($1 \leq i \leq k-1$), we try to find a vector field $\eta_i \in m_0\theta_0(2k-1)$ such that $-W_2 \frac{\partial}{\partial V_i} \neq \eta_i$ and $y^i(W_2 \circ \varphi_k) \frac{\partial}{\partial W_2} \equiv \eta_i \circ \varphi_k \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2\theta(\varphi_k)}$.

$$\begin{aligned} & y^i(W_2 \circ \varphi_k) \frac{\partial}{\partial W_2} \\ = & \left(\sum_{j=1}^{k-1} v_j y^{i+j} \right) \frac{\partial}{\partial W_2} \\ \equiv & \begin{cases} -\sum_{j=1}^{k-1-i} (V_j \circ \varphi_k) \frac{\partial}{\partial V_{i+j}} \\ \quad \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2\theta(\varphi_k)} & (1 \leq i \leq k-2), \\ 0 & \pmod{T\mathcal{R}_e(\varphi_k) + \varphi_k^2 m_0^2\theta(\varphi_k)} & (i = k-1). \end{cases} \end{aligned}$$

Thus, it follows that for any i ($1 \leq i \leq k-2$) there exists a liftable vector field over φ_k having the following form where $\tilde{\eta}_{2k-1+i} \in m_0^2\theta_0(p)$,

$$\eta_{2k-1+i} + \tilde{\eta}_{2k-1+i} = W_2 \frac{\partial}{\partial V_i} - \sum_{j=1}^{k-1-i} V_j \frac{\partial}{\partial V_{i+j}} + \text{higher terms;}$$

and there exists a liftable vector field over φ_k having the following form where $\tilde{\eta}_{3k-2} \in m_0^2\theta_0(p)$.

$$\eta_{3k-2} + \tilde{\eta}_{3k-2} = W_2 \frac{\partial}{\partial V_{k-1}} + \text{higher terms.}$$

Finally, set

$$\begin{aligned} \Pi = & \mathbb{K}W_1 \frac{\partial}{\partial W_1} + \mathbb{K}W_2 \frac{\partial}{\partial W_1} + \mathbb{K}W_2 \frac{\partial}{\partial W_2} \\ & + \sum_{i=1}^{k-2} \mathbb{K}W_1 \frac{\partial}{\partial U_i} + \sum_{i=1}^{k-2} \mathbb{K}W_2 \frac{\partial}{\partial U_i} + \sum_{i=1}^{k-1} \mathbb{K}W_2 \frac{\partial}{\partial V_i}. \end{aligned}$$

Then, Π is a $(3k-2)$ -dimensional \mathbb{K} -vector space. Let $\pi : \theta_0(p) \rightarrow \Pi$ be the canonical projection. Then, we see easily that $\pi(\eta_i + \tilde{\eta}_i)$ ($1 \leq i \leq 3k-2$) constitute a basis of Π . Thus, $\eta_i + \tilde{\eta}_i$ ($1 \leq i \leq 3k-2$) constitute a set of generators for the module of vector fields liftable over φ_k .

6.2. *Lift*(ψ_n) for $\psi_n(v_1, \dots, v_{n-1}, y) = (v_1, \dots, v_{n-1}, y^2, v_1 y, \dots, v_{n-1} y)$.

We let $(V_1, \dots, V_{n-1}, W, X_1, \dots, X_{n-1})$ be the standard coordinates of \mathbb{K}^{2n-1} . Since ${}_0\bar{\omega}\psi_n$ is bijective we first look for a basis of $\ker({}_1\bar{\omega}\psi_n)$. We can find out easily a basis of $\ker({}_1\bar{\omega}\psi_n)$ which is (for instance) the following:

$$\begin{aligned} & V_i \frac{\partial}{\partial V_j} + X_i \frac{\partial}{\partial X_j} + m_0^2 \theta_0(2n-1) \quad (1 \leq i, j \leq n-1), \\ & X_i \frac{\partial}{\partial V_j} + m_0^2 \theta_0(2n-1) \quad (1 \leq i, j \leq n-1), \\ & 2X_i \frac{\partial}{\partial W} + m_0^2 \theta_0(2n-1) \quad (1 \leq i \leq n-1), \\ & 2W \frac{\partial}{\partial W} + \sum_{j=1}^{n-1} X_j \frac{\partial}{\partial X_j} + m_0^2 \theta_0(2n-1). \end{aligned}$$

Since any component function of ψ_n is a monomial, we can determine easily the desired higher terms of liftable vector fields and thus we see that the following constitute a set of generators for the module of vector fields liftable over ψ_n .

$$\begin{aligned} & V_i \frac{\partial}{\partial V_j} + X_i \frac{\partial}{\partial X_j} \quad (1 \leq i, j \leq n-1), \\ & X_i \frac{\partial}{\partial V_j} + V_i W \frac{\partial}{\partial X_j} \quad (1 \leq i, j \leq n-1), \\ & 2X_i \frac{\partial}{\partial W} + \sum_{j=1}^{n-1} V_i V_j \frac{\partial}{\partial X_j} \quad (1 \leq i \leq n-1), \\ & 2W \frac{\partial}{\partial W} + \sum_{j=1}^{n-1} X_j \frac{\partial}{\partial X_j}. \end{aligned}$$

6.3. *Lift*(ϕ) for $\phi(x, y) = (x, y^2, y^3 + xy)$.

Let (x, y) , (V, W, X) be the standard coordinates of \mathbb{K}^2 and \mathbb{K}^3 respectively, and let $\phi : (\mathbb{K}^2, 0) \rightarrow (\mathbb{K}^3, 0)$ be the mono-germ defined by

$$\phi(x, y) = (x, y^2, y^3 + xy).$$

Set,

$$h(x, y) = (x + y^2, y) \quad \text{and} \quad H(V, W, X) = (V - W, W, X).$$

Then, both $h : \mathbb{K}^2 \rightarrow \mathbb{K}^2$ and $H : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ are analytic diffeomorphisms and preserve the origin. Moreover, we have the following:

$$\phi(x, y) = H \circ \psi_2 \circ h(x, y),$$

where ψ_2 is the mono-germ defined in Example 3.3. By this equality, f is \mathcal{A} -equivalent to ψ_2 . As same as ψ_2 , f is often used as the normal form of *Whitney umbrella* from \mathbb{K}^2 to \mathbb{K}^3 . By Subsection 6.2, we have the following:

$$Lift(\psi_2) = \left\langle V \frac{\partial}{\partial V} + X \frac{\partial}{\partial X}, X \frac{\partial}{\partial V} + VW \frac{\partial}{\partial X}, 2X \frac{\partial}{\partial W} + V^2 \frac{\partial}{\partial X}, 2W \frac{\partial}{\partial W} + X \frac{\partial}{\partial X} \right\rangle_{C_0}.$$

Thus, by using the following lemma, $Lift(\phi)$ can be characterized as the C_0 -module generated by the following 4 vector fields:

$$\begin{aligned} & (V + W) \frac{\partial}{\partial V} + X \frac{\partial}{\partial X}, X \frac{\partial}{\partial V} + (V + W)W \frac{\partial}{\partial X}, \\ & -2X \frac{\partial}{\partial V} + 2X \frac{\partial}{\partial W} + (V + W)^2 \frac{\partial}{\partial X}, -2W \frac{\partial}{\partial V} + 2W \frac{\partial}{\partial W} + X \frac{\partial}{\partial X}. \end{aligned}$$

Lemma 6.1. *Let S be a finite subset $\{s_1, \dots, s_r\}$ ($s_i \neq s_j$ if $i \neq j$) and let $f = \{f_1, \dots, f_r\}, g = \{g_1, \dots, g_r\} : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be two analytic multigerms. Suppose that there exist germs of analytic diffeomorphisms $h_i : (\mathbb{K}^n, s_i) \rightarrow (\mathbb{K}^n, s_i)$ ($1 \leq i \leq r$) and $H : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$ such that $g = H \circ f \circ h$, where $h : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^n, S)$ is the map-germ whose restriction to (\mathbb{K}^n, s_i) is h_i . Then, the mapping $L_{(f,g)} : Lift(f) \rightarrow Lift(g)$ defined by $L_{(f,g)}(\eta) = dH \circ \eta \circ H^{-1}$ is well-defined and bijective.*

Proof of Lemma 6.1

Let η be a liftable vector field over f . By definition, there exists $\xi \in \theta_S(n)$ such that $\eta \circ f = t f \circ \xi$. Since the equality $g = H \circ f \circ h$ holds, we have the following:

$$\eta \circ (H^{-1} \circ g \circ h^{-1}) = t(H^{-1} \circ g \circ h^{-1}) \circ \xi.$$

Hence, we have the following:

$$(dH \circ \eta \circ H^{-1}) \circ g = t g \circ (dh^{-1} \circ \xi \circ h).$$

This shows that $(dH \circ \eta \circ H^{-1})$ is a liftable vector field over g . Hence, the mapping $L_{(f,g)}$ is well-defined.

Since injectivity of $L_{(f,g)}$ is clear, it is sufficient to show that $L_{(f,g)}$ is surjective. Let $\tilde{\eta}$ be a liftable vector field of g . The above argument shows that $d(H^{-1}) \circ \tilde{\eta} \circ H$ is a liftable vector field of f . Since $L_{(f,g)}(d(H^{-1}) \circ \tilde{\eta} \circ H) = \tilde{\eta}$, it follows that $L_{(f,g)}$ is surjective. \square

6.4. $Lift(f)$ for $f(x, y) = \{(x, y^2), (x^2, y)\}$.

Let (X, Y) be the standard coordinates of \mathbb{K}^2 . Since ${}_0\overline{\omega}f$ is bijective we first look for a basis of $\ker({}_1\overline{\omega}f)$. We can find out easily a basis of $\ker({}_1\overline{\omega}f)$ which is (for instance) the following:

$$X \frac{\partial}{\partial X} + m_0^2 \theta_0(2), Y \frac{\partial}{\partial Y} + m_0^2 \theta_0(2).$$

Since any component function of f is a monomial, we can determine easily the desired higher terms of liftable vector fields and thus we see that the following constitute a set of generators for the module of vector fields liftable over f .

$$X \frac{\partial}{\partial X}, Y \frac{\partial}{\partial Y}.$$

6.5. *Lift(f) for $f(x) = \{(x^2, x^3), (x^3, x^2)\}$.*

Recall that the multigerm f of Example 3.6.2 is $f_1(x) = (x^2, x^3)$, $f_2(x) = (x^3, x^2)$. Let (X, Y) be the standard coordinates of \mathbb{K}^2 . Since ${}_1\overline{\omega}f$ is bijective we first look for a basis of $\ker({}_2\overline{\omega}f)$. We can find out easily a basis of $\ker({}_2\overline{\omega}f)$ which is (for instance) the following:

$$6XY \frac{\partial}{\partial X} + 4Y^2 \frac{\partial}{\partial Y} + m_0^3 \theta_0(2), 4X^2 \frac{\partial}{\partial X} + 6XY \frac{\partial}{\partial Y} + m_0^3 \theta_0(2).$$

Set $\xi_{1,1,1} = 3x^4 \frac{\partial}{\partial x}$, $\xi_{1,2,1} = 2x^3 \frac{\partial}{\partial x}$ and $\eta_{1,1} = 6XY \frac{\partial}{\partial X} + 4Y^2 \frac{\partial}{\partial Y}$. Then, we have the following:

$$\begin{aligned} \eta_{1,1} \circ f_1 - df_1 \circ \xi_{1,1,1} &= -5x^6 \frac{\partial}{\partial Y}, \\ \eta_{1,1} \circ f_2 - df_2 \circ \xi_{1,2,1} &= 0. \end{aligned}$$

Set $\eta_{1,2} = 5X^3 \frac{\partial}{\partial Y}$. Then we have the following:

$$(\eta_{1,1} + \eta_{1,2}) \circ f_1 - df_1 \circ \xi_{1,1,1} = 0, \quad (6.1)$$

$$(\eta_{1,1} + \eta_{1,2}) \circ f_2 - df_2 \circ \xi_{1,2,1} = 5x^9 \frac{\partial}{\partial Y}. \quad (6.2)$$

Set $\eta_{1,3} = -5XY^3 \frac{\partial}{\partial Y}$ and $\xi_{1,1,2} = -\frac{5}{3}x^9 \frac{\partial}{\partial x}$. Then we have the following:

$$\begin{aligned} (\eta_{1,1} + \eta_{1,2} + \eta_{1,3}) \circ f_1 - df_1 \circ (\xi_{1,1,1} + \xi_{1,1,2}) &= \frac{10}{3}x^{10} \frac{\partial}{\partial X}, \\ (\eta_{1,1} + \eta_{1,2} + \eta_{1,3}) \circ f_2 - df_2 \circ \xi_{1,2,1} &= 0. \end{aligned}$$

Set $\eta_{1,4} = -\frac{10}{3}X^2Y^2 \frac{\partial}{\partial X}$ and $\xi_{1,2,2} = -\frac{10}{9}x^8 \frac{\partial}{\partial x}$. Then we have the following:

$$(\eta_{1,1} + \eta_{1,2} + \eta_{1,3} + \eta_{1,4}) \circ f_1 - df_1 \circ (\xi_{1,1,1} + \xi_{1,1,2}) = 0, \quad (6.3)$$

$$(\eta_{1,1} + \eta_{1,2} + \eta_{1,3} + \eta_{1,4}) \circ f_2 - df_2 \circ (\xi_{1,2,1} + \xi_{1,2,2}) = \frac{20}{9}x^9 \frac{\partial}{\partial Y}. \quad (6.4)$$

Note that the right hand side of (5.3) (resp., the right hand side of (5.4)) is the right hand side of (5.1) (resp., the right hand side of (5.2)) multiplied by $(\frac{2}{3})^2$. Thus, the following vector field η_1 must be liftable over f .

$$\begin{aligned} \eta_1 &= \eta_{1,1} + \eta_{1,2} + \left(1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \cdots\right) (\eta_{1,3} + \eta_{1,4}) \\ &= (6XY - 6X^2Y^2) \frac{\partial}{\partial X} + (4Y^2 + 5X^3 - 9XY^3) \frac{\partial}{\partial Y}. \end{aligned}$$

Next, Set $\xi_{2,1,1} = 2x^3 \frac{\partial}{\partial x}$, $\xi_{2,2,1} = 3x^4 \frac{\partial}{\partial x}$ and $\eta_{2,1} = 4X^2 \frac{\partial}{\partial X} + 6XY \frac{\partial}{\partial Y}$. Then, we have the following:

$$\begin{aligned} \eta_{2,1} \circ f_1 - df_1 \circ \xi_{2,1,1} &= 0, \\ \eta_{2,1} \circ f_2 - df_2 \circ \xi_{2,2,1} &= -5x^6 \frac{\partial}{\partial X}. \end{aligned}$$

Set $\eta_{2,2} = 5Y^3 \frac{\partial}{\partial X}$. Then we have the following:

$$(\eta_{2,1} + \eta_{2,2}) \circ f_1 - df_1 \circ \xi_{2,1,1} = 5x^9 \frac{\partial}{\partial X}, \quad (6.5)$$

$$(\eta_{2,1} + \eta_{2,2}) \circ f_2 - df_2 \circ \xi_{2,2,1} = 0. \quad (6.6)$$

Set $\eta_{2,3} = -5X^3Y \frac{\partial}{\partial X}$ and $\xi_{2,2,2} = -\frac{5}{3}x^9 \frac{\partial}{\partial x}$. Then we have the following:

$$\begin{aligned} (\eta_{2,1} + \eta_{2,2} + \eta_{2,3}) \circ f_1 - df_1 \circ (\xi_{2,1,1}) &= 0, \\ (\eta_{2,1} + \eta_{2,2} + \eta_{2,3}) \circ f_2 - df_2 \circ (\xi_{2,2,1} + \xi_{2,2,2}) &= \frac{10}{3}x^{10} \frac{\partial}{\partial Y}. \end{aligned}$$

Set $\eta_{2,4} = -\frac{10}{3}X^2Y^2 \frac{\partial}{\partial Y}$ and $\xi_{2,1,2} = -\frac{10}{9}x^8 \frac{\partial}{\partial x}$. Then we have the following:

$$(\eta_{2,1} + \eta_{2,2} + \eta_{2,3} + \eta_{2,4}) \circ f_1 - df_1 \circ (\xi_{2,1,1} + \xi_{2,1,2}) = \frac{20}{9}x^9 \frac{\partial}{\partial X}, \quad (6.7)$$

$$(\eta_{2,1} + \eta_{2,2} + \eta_{2,3} + \eta_{2,4}) \circ f_2 - df_2 \circ (\xi_{2,2,1} + \xi_{2,2,2}) = 0. \quad (6.8)$$

Note that the right hand side of (5.7) (resp., the right hand side of (5.8)) is the right hand side of (5.5) (resp., the right hand side of (5.6)) multiplied by $(\frac{2}{3})^2$. Thus, the following vector field η_2 must be liftable over f .

$$\begin{aligned} \eta_2 &= \eta_{2,1} + \eta_{2,2} + \left(1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \cdots\right) (\eta_{2,3} + \eta_{2,4}) \\ &= (4X^2 + 5Y^3 - 9X^3Y) \frac{\partial}{\partial X} + (6XY - 6X^2Y^2) \frac{\partial}{\partial Y}. \end{aligned}$$

Therefore, the following constitute a set of generators for the module of vector fields liftable over f .

$$\begin{aligned} \eta_1 &= (6XY - 6X^2Y^2) \frac{\partial}{\partial X} + (4Y^2 + 5X^3 - 9XY^3) \frac{\partial}{\partial Y}, \\ \eta_2 &= (4X^2 + 5Y^3 - 9X^3Y) \frac{\partial}{\partial X} + (6XY - 6X^2Y^2) \frac{\partial}{\partial Y}. \end{aligned}$$

6.6. *Lift(f)* for $f(y) = (y^2, 0)$.

Let $f : (\mathbb{K}, 0) \rightarrow (\mathbb{K}^2, 0)$ be the mono-germ defined by $f(y) = (y^2, 0)$. As an application of Theorem 2, we obtain all liftable vector fields over f .

Let (Y, U) be the standard coordinates of the target space of f . It is easy to see the following:

$$\theta_S(f) = TK_e(f) + \mathbb{K}^2 + y \frac{\partial}{\partial U}$$

Since $\dim_{\mathbb{K}} \theta_S(f) / (TK_e(f) + \mathbb{K}^2) = 1$, by Mather's constructing method of stable mono-germs ([15]), the mono-germ $F(x, y) = (x, y^2, xy)$ is a one-parameter stable unfolding of f . Notice that F is exactly the same as the mono-germ ψ_2 defined in Subsection 6.2. Let (X, Y, U) be the standard coordinates of the target space of F . Let $g : (\mathbb{K} \times \mathbb{K}^2, (0, 0)) \rightarrow (\mathbb{K} \times \mathbb{K}^2, (0, 0))$ be defined by $g(x, y, u) = (x^2, y, u)$. Then, $Lift(g) = \langle X \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial U} \rangle_{C_0}$. Set $\tilde{\eta}_1 = X \frac{\partial}{\partial X} + U \frac{\partial}{\partial U}$, $\tilde{\eta}_2 = U \frac{\partial}{\partial X} + XY \frac{\partial}{\partial U}$, $\tilde{\eta}_3 = 2U \frac{\partial}{\partial Y} + X^2 \frac{\partial}{\partial U}$ and $\tilde{\eta}_4 = 2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U}$. By Subsection 6.2, we have the

following:

$$\begin{aligned} Lift(F) &= \langle \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4 \rangle_{C_0} \\ &= \left\{ \tilde{\alpha}_1 \left(X \frac{\partial}{\partial X} + U \frac{\partial}{\partial U} \right) + \tilde{\alpha}_2 \left(U \frac{\partial}{\partial X} + XY \frac{\partial}{\partial U} \right) \right. \\ &\quad \left. + \tilde{\alpha}_3 \left(2U \frac{\partial}{\partial Y} + X^2 \frac{\partial}{\partial U} \right) + \tilde{\alpha}_4 \left(2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U} \right) \right\}, \end{aligned}$$

where $\tilde{\alpha}_i$ ($1 \leq i \leq 4$) are analytic function-germs of three variables X, Y, U . Thus, we have the following:

$$\begin{aligned} &Lift(F) \cap Lift(g) \\ &= \left\{ \tilde{\alpha}_1 \left(X \frac{\partial}{\partial X} + U \frac{\partial}{\partial U} \right) + \tilde{\alpha}_2 \left(U \frac{\partial}{\partial X} + XY \frac{\partial}{\partial U} \right) + \tilde{\alpha}_3 \left(2U \frac{\partial}{\partial Y} + X^2 \frac{\partial}{\partial U} \right) \right. \\ &\quad \left. + \tilde{\alpha}_4 \left(2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U} \right) \middle| \tilde{\alpha}_1 X + \tilde{\alpha}_2 U \text{ can be divided by } X \right\} \\ &= \left\{ \tilde{\alpha}_1 \left(X \frac{\partial}{\partial X} + U \frac{\partial}{\partial U} \right) + \tilde{\alpha}_2 \left(U \frac{\partial}{\partial X} + XY \frac{\partial}{\partial U} \right) + \tilde{\alpha}_3 \left(2U \frac{\partial}{\partial Y} + X^2 \frac{\partial}{\partial U} \right) \right. \\ &\quad \left. + \tilde{\alpha}_4 \left(2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U} \right) \middle| \tilde{\alpha}_2 \text{ can be divided by } X \right\}. \end{aligned}$$

Define $\alpha_i : (\mathbb{K}^2, 0) \rightarrow \mathbb{K}$ ($1 \leq i \leq 4$) by $\alpha_i(Y, U) = \tilde{\alpha}_i(0, Y, U)$. Then, by Theorem 2, we have the following:

$$\begin{aligned} &Lift(f) \\ &= \left\{ \alpha_1 U \frac{\partial}{\partial U} + 2\alpha_3 U \frac{\partial}{\partial Y} + \alpha_4 \left(2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U} \right) \right\} \\ &= \left\langle U \frac{\partial}{\partial U}, U \frac{\partial}{\partial Y}, Y \frac{\partial}{\partial Y} \right\rangle_{C_0}. \end{aligned}$$

Since three vector fields $U \frac{\partial}{\partial U}, U \frac{\partial}{\partial Y}, Y \frac{\partial}{\partial Y}$ are linearly independent, the minimal number of generators for $Lift(f)$ is 3, which is strictly greater than the dimension of the target space of f .

6.7. $Lift(f_k)$ for $f_k(y) = (y^2, y^{2k+1})$ ($k \geq 1$).

Let $f_k : (\mathbb{K}, 0) \rightarrow (\mathbb{K}^2, 0)$ ($k \geq 1$) be the mono-germ defined by $f_k(y) = (y^2, y^{2k+1})$. As an application of Theorem 2, we obtain all liftable vector fields over f_k .

Let (Y, U) be the standard coordinates of the target space of f_k . It is easy to see the following:

$$\theta_S(f_k) = TK_e(f_k) + \mathbb{K}^2 + y \frac{\partial}{\partial U}$$

Since $\dim_{\mathbb{K}} \theta_S(f_k) / (TK_e(f_k) + \mathbb{K}^2) = 1$, by Mather's constructing method of stable mono-germs ([15]), the mono-germ $F_k(x, y) = (x, y^2, y^{2k+1} + xy)$ is a one-parameter stable unfolding of f_k . Set $F(x, y) = (x, y^2, xy)$. Let (X, Y, U) be the standard coordinates of the target space of F_k . Set $h_k(x, y) = (x + y^{2k}, y)$ and $H_k(X, Y, U) = (X - Y^k, Y, U)$. Then, both h_k and H_k are analytic diffeomorphisms preserving the origin, and we have that $F_k = H_k \circ F \circ h_k$. Set $\tilde{\eta}_1 = X \frac{\partial}{\partial X} + U \frac{\partial}{\partial U}$, $\tilde{\eta}_2 = U \frac{\partial}{\partial X} + XY \frac{\partial}{\partial U}$, $\tilde{\eta}_3 = 2U \frac{\partial}{\partial Y} + X^2 \frac{\partial}{\partial U}$ and $\tilde{\eta}_4 = 2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U}$. By Subsection 6.2, we have the following:

$$Lift(F) = \langle \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4 \rangle_{C_0}.$$

Set $\eta_1 = dH_k \circ \tilde{\eta}_1 \circ H_k^{-1}$, $\eta_2 = dH_k \circ \tilde{\eta}_2 \circ H_k^{-1}$, $\eta_3 = dH_k \circ \tilde{\eta}_3 \circ H_k^{-1}$ and $\eta_4 = dH_k \circ \tilde{\eta}_4 \circ H_k^{-1}$. By Lemma 6.1, we have the following:

$$Lift(F_k) = \langle \eta_1, \eta_2, \eta_3, \eta_4 \rangle_{C_0}.$$

By calculations, we have the following:

$$\begin{cases} \eta_1(X, Y, U) &= (X + Y^k) \frac{\partial}{\partial X} + U \frac{\partial}{\partial U}, \\ \eta_2(X, Y, U) &= U \frac{\partial}{\partial X} + (X + Y^k) Y \frac{\partial}{\partial U}, \\ \eta_3(X, Y, U) &= -2kY^{k-1}U \frac{\partial}{\partial X} + 2U \frac{\partial}{\partial Y} + (X + Y^k)^2 \frac{\partial}{\partial U}, \\ \eta_4(X, Y, U) &= -2kY^k \frac{\partial}{\partial X} + 2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U}. \end{cases}$$

Let $g : (\mathbb{K} \times \mathbb{K}^2, (0, 0)) \rightarrow (\mathbb{K} \times \mathbb{K}^2, (0, 0))$ be defined by $g(x, y, u) = (x^2, y, u)$. Then, $Lift(g) = \langle X \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial U} \rangle_{C_0}$. Thus, we have the following:

$$\begin{aligned} & Lift(F_k) \cap Lift(g) \\ &= \left\{ \sum_{i=1}^4 \tilde{\alpha}_i \eta_i \mid \tilde{\alpha}_1(X + Y^k) + \tilde{\alpha}_2 U - 2k\tilde{\alpha}_3 Y^{k-1}U - 2k\tilde{\alpha}_4 Y^k \text{ can be divided by } X \right\}, \end{aligned}$$

where $\tilde{\alpha}_i$ ($1 \leq i \leq 4$) are analytic function-germs of three variables X, Y, U . Define $\alpha_i : (\mathbb{K}^2, 0) \rightarrow \mathbb{K}$ ($1 \leq i \leq 4$) by $\alpha_i(Y, U) = \tilde{\alpha}_i(0, Y, U)$. Then, by Theorem 2, $Lift(f_k)$ can be characterized as follows:

$$\begin{aligned} & Lift(f_k) \\ &= \left\{ 2(U\alpha_3 + Y\alpha_4) \frac{\partial}{\partial Y} \right. \\ & \quad \left. + (U(\alpha_1 + \alpha_4) + Y^{2k}\alpha_3 + Y^{k+1}\alpha_2) \frac{\partial}{\partial U} \mid Y^k\alpha_1 + U\alpha_2 - 2kY^{k-1}U\alpha_3 - 2kY^k\alpha_4 = 0 \right\} \\ &= \left\{ 2(U\alpha_3 + Y\alpha_4) \frac{\partial}{\partial Y} \right. \\ & \quad \left. + (U(\alpha_1 + \alpha_4) + Y^{2k}\alpha_3 + Y^{k+1}\alpha_2) \frac{\partial}{\partial U} \mid Y^k(\alpha_1 - 2k\alpha_4) + U(\alpha_2 - 2kY^{k-1}\alpha_3) = 0 \right\} \\ &= \left\{ 2(U\alpha_3 + Y\alpha_4) \frac{\partial}{\partial Y} \right. \\ & \quad \left. + (U(\alpha_1 + \alpha_4) + Y^{2k}\alpha_3 + Y^{k+1}\alpha_2) \frac{\partial}{\partial U} \mid \alpha_1 - 2k\alpha_4 = U\beta, \alpha_2 - 2kY^{k-1}\alpha_3 = -Y^k\beta \right\} \\ &= \left\{ 2(U\alpha_3 + Y\alpha_4) \frac{\partial}{\partial Y} + (U(1 + 2k)\alpha_4 + U^2\beta) + Y^{2k}(1 + 2k)\alpha_3 - Y^{2k+1}\beta \right\} \frac{\partial}{\partial U} \\ &= \left\{ \alpha_3 \left(2U \frac{\partial}{\partial Y} + (1 + 2k)Y^{2k} \frac{\partial}{\partial U} \right) \right. \\ & \quad \left. + \alpha_4 \left(2Y \frac{\partial}{\partial Y} + (1 + 2k)U \frac{\partial}{\partial U} \right) + \beta (U^2 - Y^{2k+1}) \frac{\partial}{\partial U} \right\} \\ &= \left\{ \left(\alpha_3 - \frac{1}{1 + 2k}Y\beta \right) \left(2U \frac{\partial}{\partial Y} + (1 + 2k)Y^{2k} \frac{\partial}{\partial U} \right) \right. \\ & \quad \left. + \left(\alpha_4 + \frac{1}{1 + 2k}U\beta \right) \left(2Y \frac{\partial}{\partial Y} + (1 + 2k)U \frac{\partial}{\partial U} \right) \right\} \\ &= \left\langle 2U \frac{\partial}{\partial Y} + (1 + 2k)Y^{2k} \frac{\partial}{\partial U}, 2Y \frac{\partial}{\partial Y} + (1 + 2k)U \frac{\partial}{\partial U} \right\rangle_{C_0}, \end{aligned}$$

where β is an analytic function-germ of two variables Y, U .

6.8. *Lift(S_k^\pm) for $S_k^\pm(x, y) = (x, y^2, y^3 \pm x^{k+1}y)$ ($k \geq 0$).*

Let $S_k^\pm : (\mathbb{K}^2, 0) \rightarrow (\mathbb{K}^3, 0)$ be the mono-germ defined by $S_k^\pm(x, y) = (x, y^2, y^3 \pm x^{k+1}y)$ ($k \geq 0$). The mono-germ S_k^\pm can be found in the classification list of \mathcal{A} -simple mono-germ from the plane to 3-space due to Mond ([19]). Here, a multigerms $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ is said to be \mathcal{A} -simple if there exists a finite number of \mathcal{A} -equivalence classes such that for any positive integer d and any analytic mapping $F : U \rightarrow V$ where $U \subset \mathbb{K}^n \times \mathbb{K}^d$ is a neighbourhood of $S \times 0$, $V \subset \mathbb{K}^p \times \mathbb{K}^d$ is a neighbourhood of $(0, 0)$, $F(x, \lambda) = (f_\lambda(x), \lambda)$ and the germ of f_0 at S is f , there exists a sufficiently small neighbourhood $W_i \subset U$ of $(s_i, 0)$ ($1 \leq i \leq |S|$) such that for every $\{(x_1, \lambda), \dots, (x_r, \lambda)\}$ ($r \leq |S|$) with $(x_i, \lambda) \in W_i$ and $F(x_1, \lambda) = \dots = F(x_r, \lambda)$ the multigerms $f_\lambda : (\mathbb{K}^n, \{x_1, \dots, x_r\}) \rightarrow (\mathbb{K}^p, f_\lambda(x_i))$ lies in one of these finite \mathcal{A} -equivalence classes. As an application of Theorem 2, we obtain all liftable vector fields over S_k^\pm .

Let (X, Y, U) be the standard coordinates of the target space of S_k^\pm . It is easy to see the following:

$$\theta_S(S_k^\pm) = T\mathcal{K}_e(S_k^\pm) + \mathbb{K}^3 + y \frac{\partial}{\partial U}$$

Since $\dim_{\mathbb{K}}(\theta_S(S_k^\pm)/T\mathcal{K}_e(S_k^\pm) + \mathbb{K}^3) \leq 1$, by Mather's constructing method of stable mono-germs ([15]), the mono-germ $F_k^\pm(x, y, u) = (x, y^2, y^3 \pm x^{k+1}y + uy, u)$ is a one-parameter stable unfolding of S_k^\pm . Set $F(x, y, u) = (x, y^2, y^3 + uy, u)$. Let (X, Y, U, V) be the standard coordinates of the target space of F_k . Set $h_k^\pm(x, y, u) = (x, y, u \pm x^{k+1})$ and $H_k^\pm(X, Y, U, V) = (X, Y, U, V \mp X^{k+1})$. Then, both h_k^\pm and H_k^\pm are analytic diffeomorphisms preserving the origin, and we have that $F_k^\pm = H_k^\pm \circ F \circ h_k^\pm$. Set $\tilde{\eta}_1 = U \frac{\partial}{\partial U} + (Y + V) \frac{\partial}{\partial V}$, $\tilde{\eta}_2 = (Y + V)Y \frac{\partial}{\partial U} + U \frac{\partial}{\partial V}$, $\tilde{\eta}_3 = 2U \frac{\partial}{\partial Y} + (Y + V)^2 \frac{\partial}{\partial U} - 2U \frac{\partial}{\partial V}$, $\tilde{\eta}_4 = 2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U} - 2Y \frac{\partial}{\partial V}$ and $\tilde{\eta}_5 = \frac{\partial}{\partial X}$. By calculations, we have the following:

$$\text{Lift}(F) = \langle \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4, \tilde{\eta}_5 \rangle_{C_0}.$$

Set $\eta_1 = d(H_k^\pm) \circ \tilde{\eta}_1 \circ (H_k^\pm)^{-1}$, $\eta_2 = d(H_k^\pm) \circ \tilde{\eta}_2 \circ (H_k^\pm)^{-1}$, $\eta_3 = d(H_k^\pm) \circ \tilde{\eta}_3 \circ (H_k^\pm)^{-1}$, $\eta_4 = d(H_k^\pm) \circ \tilde{\eta}_4 \circ (H_k^\pm)^{-1}$ and $\eta_5 = d(H_k^\pm) \circ \tilde{\eta}_5 \circ (H_k^\pm)^{-1}$. By Lemma 6.1, we have the following:

$$\text{Lift}(F_k) = \langle \eta_1, \eta_2, \eta_3, \eta_4, \eta_5 \rangle_{C_0}.$$

By calculations, we have the following:

$$\begin{cases} \eta_1(X, Y, U, V) &= U \frac{\partial}{\partial U} + (Y + V \pm X^{k+1}) \frac{\partial}{\partial V}, \\ \eta_2(X, Y, U, V) &= Y(Y + V \pm X^{k+1}) \frac{\partial}{\partial U} + U \frac{\partial}{\partial V}, \\ \eta_3(X, Y, U, V) &= 2U \frac{\partial}{\partial Y} + (Y + V \pm X^{k+1})^2 \frac{\partial}{\partial U} - 2U \frac{\partial}{\partial V}, \\ \eta_4(X, Y, U, V) &= 2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U} - 2Y \frac{\partial}{\partial V}, \\ \eta_5(X, Y, U, V) &= \frac{\partial}{\partial X} \mp (k+1)X^k \frac{\partial}{\partial V}. \end{cases}$$

Let $g : (\mathbb{K}^3 \times \mathbb{K}, (0, 0)) \rightarrow (\mathbb{K}^3 \times \mathbb{K}, (0, 0))$ be defined by $g(x, y, u, v) = (x, y, u, v^2)$. Then, $\text{Lift}(g) = \langle \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial U}, V \frac{\partial}{\partial V} \rangle_{C_0}$. Thus, we have the following:

$$\text{Lift}(F_k) \cap \text{Lift}(g) = \left\{ \sum_{i=1}^5 \tilde{\alpha}_i \eta_i \mid \Phi(X, Y, U, V) \text{ can be divided by } V \right\},$$

where $\tilde{\alpha}_i$ ($1 \leq i \leq 5$) are analytic function-germs of four variables X, Y, U, V and $\Phi(X, Y, U, V)$ is given as follows:

$$\begin{aligned}\Phi(X, Y, U, V) &= \tilde{\alpha}_1(X, Y, U, V) (Y + V \pm X^{k+1}) \\ &\quad + \tilde{\alpha}_2(X, Y, U, V)U - 2\tilde{\alpha}_3(X, Y, U, V)U \\ &\quad - 2\tilde{\alpha}_4(X, Y, U, V)Y \mp (k+1)\tilde{\alpha}_5(X, Y, U, V)X^k.\end{aligned}$$

Define $\alpha_i : (\mathbb{K}^3, 0) \rightarrow \mathbb{K}$ ($1 \leq i \leq 5$) by $\alpha_i(X, Y, U) = \tilde{\alpha}_i(X, Y, U, 0)$. Then, by Theorem 2, any element of $Lift(S_k^\pm)$ has the following form:

$$\begin{aligned}\alpha_1 U \frac{\partial}{\partial U} + \alpha_2 Y (Y \pm X^{k+1}) \frac{\partial}{\partial Y} + \alpha_3 \left(2U \frac{\partial}{\partial Y} + (Y \pm X^{k+1})^2 \frac{\partial}{\partial U} \right) \\ + \alpha_4 \left(2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U} \right) + \alpha_5 \frac{\partial}{\partial X}.\end{aligned}$$

And, by Theorem 2 again, the unique restriction on α_i ($1 \leq i \leq 5$) is as follows.

Condition 6.1.

$$\Phi(X, Y, U, 0) = \alpha_1 (Y \pm X^{k+1}) + \alpha_2 U - 2\alpha_3 U - 2\alpha_4 Y \mp (k+1)\alpha_5 X^k = 0.$$

In the case $k = 0$, by Condition 6.1 it follows that α_5 can be expressed by using α_i ($1 \leq i \leq 4$). Thus, it is easy to obtain four vector fields which constitute a generators of $Lift(S_0^\pm)$. In the case $k \geq 1$, by Condition 6.1, we have the following expressions:

$$\begin{aligned}\alpha_1 - 2\alpha_4 &= \beta_1 X^k + \gamma U, \\ \alpha_2 - 2\alpha_3 &= \beta_2 X^k - \gamma Y,\end{aligned}$$

where β_1, β_2, γ are analytic function-germs $(\mathbb{K}^3, 0) \rightarrow \mathbb{K}$. Therefore, we have the following:

$$\alpha_5 = \pm \frac{1}{k+1} (\beta_1 Y + \beta_2 U \pm (2\alpha_4 + \beta_1 X^k + \gamma U) X).$$

Hence, $Lift(S_k^\pm)$ in the case $k \geq 1$ can be characterized as follows:

$$\begin{aligned}& Lift(S_k^\pm) \\ &= \left\{ (2\alpha_4 + \beta_1 X^k + \gamma U) U \frac{\partial}{\partial U} + (2\alpha_3 + \beta_2 X^k - \gamma Y) (Y \pm X^{k+1}) Y \frac{\partial}{\partial U} \right. \\ &\quad + \alpha_3 \left(2U \frac{\partial}{\partial Y} + (Y \pm X^{k+1})^2 \frac{\partial}{\partial U} \right) + \alpha_4 \left(2Y \frac{\partial}{\partial Y} + U \frac{\partial}{\partial U} \right) \\ &\quad \left. \pm \frac{1}{k+1} (\beta_1 Y + \beta_2 U \pm (2\alpha_4 + \beta_1 X^k + \gamma U) X) \frac{\partial}{\partial X} \right\} \\ &= \left\langle 2U \frac{\partial}{\partial Y} + (3Y^2 \pm 4X^{k+1}Y + X^{2k+2}) \frac{\partial}{\partial U}, \frac{2X}{k+1} \frac{\partial}{\partial X} + 2Y \frac{\partial}{\partial Y} + 3U \frac{\partial}{\partial U}, \right. \\ &\quad \left. \pm \frac{1}{k+1} (Y \pm X^{k+1}) \frac{\partial}{\partial X} + X^k U \frac{\partial}{\partial U}, \pm \frac{U}{k+1} \frac{\partial}{\partial X} + X^k Y (Y \pm X^{k+1}) \frac{\partial}{\partial U}, \right. \\ &\quad \left. + \frac{XU}{k+1} \frac{\partial}{\partial X} + (U^2 - Y^2 (Y \pm X^{k+1})) \frac{\partial}{\partial U} \right\rangle_{C_0}.\end{aligned}$$

Set $\mathbf{v}_1 = 2U \frac{\partial}{\partial Y} + (3Y^2 \pm 4X^{k+1}Y + X^{2k+2}) \frac{\partial}{\partial U}$, $\mathbf{v}_2 = \frac{2X}{k+1} \frac{\partial}{\partial X} + 2Y \frac{\partial}{\partial Y} + 3U \frac{\partial}{\partial U}$, $\mathbf{v}_3 = \pm \frac{1}{k+1} (Y \pm X^{k+1}) \frac{\partial}{\partial X} + X^k U \frac{\partial}{\partial U}$, $\mathbf{v}_4 = \pm \frac{U}{k+1} \frac{\partial}{\partial X} + X^k Y (Y \pm X^{k+1}) \frac{\partial}{\partial U}$ and $\mathbf{v}_5 = \frac{XU}{k+1} \frac{\partial}{\partial X} + (U^2 - Y^2 (Y \pm X^{k+1})) \frac{\partial}{\partial U}$. Then, we have the following relation:

$$-Y\mathbf{v}_1 + U\mathbf{v}_2 \pm X\mathbf{v}_4 = \mathbf{v}_5.$$

And, it is easily seen that none of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ can be generated by others.

It is also easily seen that the minimal number of generators for $Lift(S_k^\pm)$ is less than or equal to the minimal number of generators for $Lift(F)$. Thus, the minimal number of generators for $Lift(S_k^\pm)$ is less than or equal to 5. It is interesting to observe that the minimal number of generators for $Lift(S_k^\pm)$ is always 4. It is also interesting to observe that the germs B_k^\pm , C_k^\pm and F_4 in Mond's classification ([19]) also have less than or equal to 5 generators in the set of liftable vector fields, since they all admit one-parameter stable unfoldings \mathcal{A} -equivalent to F .

6.9. *Lift(f) for $f(x, y) = \{(x, y^3 + xy), (x, y^2)\}$.*

Let $f = \{f_1, f_2\}$ be the plane to plane bigerm defined by $f_1(x, y) = (x, y^3 + xy)$ and $f_2(x, y) = (x, y^2)$. Consider the one-parameter stable unfolding $F = \{F_1, F_2\}$ defined by

$$\begin{cases} (x, y^3 + xy, z) \\ (x, y^2 + z, z) \end{cases}.$$

It is not hard to see that $Lift(F_1) = \langle 2X \frac{\partial}{\partial X} + 3Y \frac{\partial}{\partial Y}, 9Y \frac{\partial}{\partial X} - 2X^2 \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} \rangle_{C_0}$ and $Lift(F_2) = \langle \frac{\partial}{\partial X}, Y \frac{\partial}{\partial Y} + Z \frac{\partial}{\partial Z}, \frac{\partial}{\partial Y} + \frac{\partial}{\partial Z} \rangle_{C_0}$ where (X, Y, Z) are the variables in the target. So $Lift(F) = Lift(F_1) \cap Lift(F_2) =$

$$\langle (Z - Y) \frac{\partial}{\partial Z}, 2X \frac{\partial}{\partial X} + 3Y \frac{\partial}{\partial Y} + 3Z \frac{\partial}{\partial Z}, 9Y \frac{\partial}{\partial X} - 2X^2 \frac{\partial}{\partial Y} - 2X^2 \frac{\partial}{\partial Z} \rangle_{C_0}.$$

To apply Theorem 2, we consider $g(x, y, z) = (x, y, z^2)$. Then $Lift(g) = \langle \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, Z \frac{\partial}{\partial Z} \rangle_{C_0}$. So $Lift(F) \cap Lift(g) =$

$$\begin{aligned} & \langle 2X \frac{\partial}{\partial X} + 3Y \frac{\partial}{\partial Y} + 3Z \frac{\partial}{\partial Z}, 9Y^2 \frac{\partial}{\partial X} - 2X^2 Y \frac{\partial}{\partial Y} - 2X^2 Z \frac{\partial}{\partial Z}, \\ & (27YZ + 4X^3) \frac{\partial}{\partial X} + (-6X^2 Z + 6X^2 Y) \frac{\partial}{\partial Y}, (Z^2 - YZ) \frac{\partial}{\partial Z} \rangle_{C_0}. \end{aligned}$$

And finally $Lift(f) = \langle 2X \frac{\partial}{\partial X} + 3Y \frac{\partial}{\partial Y}, 9Y^2 \frac{\partial}{\partial X} - 2X^2 Y \frac{\partial}{\partial Y} \rangle_{C_0}$.

Remark 1. *Few complete classifications of simple multigerms are known, namely [12], [32], [8] and [23]. Based on these classifications, it seems that most simple germs, except for a few cases which can be excluded by the multiplicity ([21]), admit one-parameter stable unfoldings. This suggests that the method followed above can be applied to most simple germs.*

7. THE CASE $n > p$

Let $\bar{f} : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ be a corank 1 simple germ with $n > p$. By [25], \bar{f} is \mathcal{A} -equivalent to

$$f(x_1, \dots, x_p, \dots, x_n) = (x_1, \dots, x_{p-1}, g(x_1, \dots, x_p) + \sum_{j=p+1}^n a_j x_j^2),$$

where $a_j = \pm 1$, $(p+1 \leq j \leq n)$. Let $f_0 : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0)$ be the germ such that $f_0(x_1, \dots, x_p) = (x_1, \dots, x_{p-1}, g(x_1, \dots, x_p))$.

Proposition 8. $Lift(f) = Lift(f_0)$.

Proof of Proposition 8

First suppose $\eta \in Lift(f)$, by definition there exists $\xi \in \theta_0(n)$ such that $\eta \circ f(x_1, \dots, x_n) = df \circ \xi(x_1, \dots, x_n)$. In particular, $\eta \circ f(x_1, \dots, x_p, 0, \dots, 0) = df \circ \xi(x_1, \dots, x_p, 0, \dots, 0)$ and therefore $\eta \circ f_0(x_1, \dots, x_p) = df_0 \circ \bar{\xi}(x_1, \dots, x_p)$ where

$$\bar{\xi}(x_1, \dots, x_p) = \begin{pmatrix} \xi_1(x_1, \dots, x_p, 0, \dots, 0) \\ \vdots \\ \xi_p(x_1, \dots, x_p, 0, \dots, 0) \end{pmatrix}.$$

Therefore, $\eta \in Lift(f_0)$ and so $Lift(f) \subset Lift(f_0)$.

Now suppose $\eta_0 \in Lift(f_0)$, by definition there exists $\xi_0 \in \theta_0(p)$ such that $\eta_0 \circ f_0 = df_0 \circ \xi_0$. Note that $df_0 \circ \xi_0 = df \circ \xi$ where

$$\xi(x_1, \dots, x_n) = \begin{pmatrix} \xi_{0_1}(x_1, \dots, x_p) \\ \vdots \\ \xi_{0_p}(x_1, \dots, x_p) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We have that $\eta_0 \circ f(x_1, \dots, x_n) =$

$$\begin{pmatrix} \eta_{0_1} \circ f(x_1, \dots, x_p, 0, \dots, 0) \\ \vdots \\ \eta_{0_p} \circ f(x_1, \dots, x_p, 0, \dots, 0) \end{pmatrix} + \begin{pmatrix} \eta_{0_1} \circ f(x_1, \dots, x_n) - \eta_{0_1} \circ f(x_1, \dots, x_p, 0, \dots, 0) \\ \vdots \\ \eta_{0_p} \circ f(x_1, \dots, x_n) - \eta_{0_p} \circ f(x_1, \dots, x_p, 0, \dots, 0) \end{pmatrix}$$

The first matrix is equal to $\eta_0 \circ f_0 = df_0 \circ \xi_0 = df \circ \xi$ and, by Hadamard's Lemma ([7]), there exist functions $\bar{\xi}_{p+1}, \dots, \bar{\xi}_n$ such that the second matrix is equal to

$$df \circ \begin{pmatrix} \eta_{0_1} \circ f(x_1, \dots, x_n) - \eta_{0_1} \circ f(x_1, \dots, x_p, 0, \dots, 0) \\ \vdots \\ \eta_{0_p} \circ f(x_1, \dots, x_n) - \eta_{0_p} \circ f(x_1, \dots, x_p, 0, \dots, 0) \\ 0 \\ \bar{\xi}_{p+1} \\ \vdots \\ \bar{\xi}_n \end{pmatrix}.$$

So $\eta_0 \in Lift(f)$ and the proposition is proved. Q.E.D.

The proposition holds for multigerms too since the above proof can be repeated for each branch. Thus, we can obtain the following

Example 7.1. Let $f = \{f_1, f_2\} : (\mathbb{K}^n, \{0, 0\}) \rightarrow (\mathbb{K}^2, 0)$, $n > 2$, be the bigerm defined by

$$\begin{cases} f_1(x, y, u_1, \dots, u_{n-2}) = (x, y^3 + xy + \sum_{i=1}^{n-2} a_i u_i^2) \\ f_2(x, y, u_1, \dots, u_{n-2}) = (x, y^2 + \sum_{i=1}^{n-2} b_i u_i^2) \end{cases}$$

where $a_i = \pm 1$ and $b_i = \pm 1$, ($1 \leq i \leq n-2$). Then $Lift(f) = \langle 2X \frac{\partial}{\partial X} + 3Y \frac{\partial}{\partial Y}, 9Y^2 \frac{\partial}{\partial X} + 2X^2Y \frac{\partial}{\partial Y} \rangle_{C_0}$.

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